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MATHEMATICS

On Projective Semimodules

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1 Introduction

Numerous investigations on freeness of projective modules over rings have led to many remarkable results. It suffices to mention the Quilen-Suslin theorem confirming Serre's famous conjecture on coincidence of the classes of free and projective modules over polynomial rings with coefficients in a field. At the same time there are only few results which deal with the problem on freeness of projective semimodules over semirings. In [7] O. Sokratova proved that for any nonzero commutative, additively idempotent semiring S, free S-semimodules constitute a proper subclass of the class of projective S-semimodules. Later Y.Katsov [4] extended this result to additively regular semirings with non-empty sets of characters. As a consequence of the latter, he showed that the classes of projective and free semimodules over the polynomial semiring $R[x_1, x_2, \ldots, x_n]$ over an additively regular division semiring R coincide if and only if R is a field. In Section 3 of this work the proofs of the Katsov's results are presented.

Next, It is known that all projective semimodules over the semiring $\mathbf{N}$ of nonnegative integers, i.e., all projective abelian monoids are free. Recently, in [5], A. Patchkoria introduced semirings with valuations in nonnegative integers and proved that all projective semimodules over them are free. Among other consequences of this theorem, he obtained that if E is either a group, or a submonoid of a free monoid, or a submonoid of a free abelian monoid, then the classes of projective and free semimodules over the monoid semiring of E with coefficients in $\mathbf{N}$ coincide. Section 4 is concerned with these results.

In Section 5, using the aforementioned theorem of Patchkoria, we calculate the Grothendieck group K_0R for any semiring R with valuations in nonnegative integers.

Finally, in Section 6 we give some strengthening of the main theorem of [5] about coincidence of the classes of projective and free semimodules over semirings with valuations in nonnegative integers.

2 Preliminaries

A semiring $R = (R, +, 0, \cdot, 1)$ is an algebraic structure, where $(R, +, 0)$ is an abelian monoid, $(R, \cdot, 1)$ is a monoid and

$$r \cdot (r' + r'') = r \cdot r' + r \cdot r'',$$

$$(r' + r'') \cdot r = r' \cdot r + r'' \cdot r,$$

$$r \cdot 0 = 0 \cdot r = 0$$

for all $r, r', r'' \in R$. To avoid trivial exceptions, we assume that $1 \neq 0$. A map $\varphi : R \longrightarrow R'$ between semirings R and R' is called a semiring homomorphism if $\varphi : (R, +, 0) \longrightarrow (R', +, 0)$ and $\varphi : (R, \cdot, 1) \longrightarrow (R', \cdot, 1)$ are monoid homomorphisms.

Let R be a semiring. Recall that an abelian monoid $M = (M, +, 0)$ together with a map $R \times M \longrightarrow M$, written $(r, m) \mapsto rm$, is called a (left) R-semimodule if

$$r(m + m') = rm + rm',$$

$$(r + r')m = rm + r'm,$$

$$(r \cdot r')m = r(r'm),$$

$$1m = m, \quad 0m = 0$$

for all $r, r' \in R$ and for all $m, m' \in M$. Right semimodules over R are similarly defined.

A map $f : A \longrightarrow B$ between R-semimodules A and B is called an R-homomorphism if $f(a + a') = f(a) + f(a')$ and $f(ra) = rf(a)$ for all $a, a' \in A$ and $r \in R$. It is obvious that any R-homomorphism carries 0 into 0.

A subset T of an R-semimodule A is a set of R-generators for A if every element of A can be written as a finite sum $\sum r_i t_i$, where $r_i \in R$ and $t_i \in T$. A is a free R-semimodule on T, or T is an R-basis of A, if each element a of A has a unique representation of the form $a = \sum_{t \in T} r_t t$, called the representation of a by the R-basis T, where $r_t \in R$ and all but a finite number of the r_t are zero.

Proposition 2.1. *Let R be a semiring without zero divisors and F a free R-semimodule. If $rw = 0$, $r \in R$, $w \in F$, then $r = 0$ or $w = 0$.* □

An R-semimodule P is called projective if, for each surjective R-homomorphism $\tau : B \longrightarrow C$ and each R-homomorphism $f : P \longrightarrow C$, there is an R-homomorphism $f' : P \longrightarrow B$ such that $f = \tau f'$.

Let $\mathcal{M}$ denote the variety of abelian monoids, and $\mathcal{M}_R$ and ${}_R\mathcal{M}$ be the categories of right and left semimodules, respectively, over a semiring R. The tensor product bifunctor $- \otimes - : \mathcal{M}_R \times_R \mathcal{M} \to \mathcal{M}$ on a right semimodule $A \in \mathcal{M}_R$ and a left semimodules $B \in_R \mathcal{M}$ can be described as the factor monoid F/σ of the free abelian monoid $F \in \mathcal{M}$, generated by the cartesian product $A \times B$, factorized with respect to the congruance σ on F generated by all ordered pairs having the form

$$\langle (a_1 + a_2, b), (a_1, b) + (a_2, b) \rangle, \quad \langle (a, b_1 + b_2), (a, b_1) + (a, b_2) \rangle$$

and

$$\langle (ar, b), (a, rb) \rangle, \quad \text{with} \quad a_1, a_2 \in A, \quad b_1, b_2 \in B \quad \text{and} \quad r \in R.$$

Thus, $A \otimes_R B = F/\sigma$, $u\omega = f : A \times B \longrightarrow A \otimes_R B = F/\sigma$ (where ω is the canonical inclusion of $A \times B$ into F, and $u : F \longrightarrow F/\sigma$ the canonical epimorphism) is an initial object in the category $\mathrm{bih}(A, B)$ of bihomomorphisms from $A \times B$; and $A \otimes_R B$ is generated by the elements $f(a \times b) \stackrel{def}{=} a \otimes b$ with $a \in A$ and $b \in B$.

Now, suppose that R is a semiring and M an arbitrary, multiplicatively written, monoid. The free R-semimodule $R[M]$ generated by the elements $x \in M$ consists of the finite sums $\sum_{x \in M} r_x x$ with coefficients $r_x \in R$. The product in M induces a product

$$\sum_{x \in M} r_x x \cdot \sum_{y \in M} r'_y y = \sum_{x,y \in M} (r_x r'_y) xy$$

of two such elements, and makes $R[M]$ a semiring, called the monoid semiring of M with coefficients in the semiring R.

Next, we say that an element m of monoid M is regular if $m = mxm$ for some $x \in M$; M is regular if all its elements are regular. If S is a monoid and for some $a, b \in S$ we have $a = aba$ and $b = bab$, then we say that b is an inverse of a. A monoid where every element has a unique inverse is an inverse monoid. For an abelian monoid M both notions coincide, *i.e.*, M is regular iff it is inverse ([1, Theorem 1.17]).

Recall that due to [1, Theorem 4.11] (see also [6, Theorem II.2.6]) each additive abelian inverse monoid $M = (M, +, 0)$ is isomorphic to its Clifford representation $R = [Y; G_\alpha, \varphi_{\alpha,\beta}]$, where Y is semilattice, G_α is an abelian group for each $\alpha \in Y$, and for each pair $\alpha, \beta \in Y \alpha \leq \beta, \varphi_{\alpha,\beta} : G_\alpha \rightarrow G_\beta$ are group homomorphisms. All homomorphisms of abelian Clifford monoids $[Y; G_\alpha, \varphi_{\alpha,\beta}]$ are described by the following proposition.

Proposition 2.2. ([6, Proposition II.2.8]) *Consider the Clifford monoids $R = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ and $S = [Z; H_\alpha, \psi_{\alpha,\beta}]$. Let $\eta : Y \longrightarrow Z$ be a homomorphism, and for each $\alpha \in Y$, let $\chi_\alpha : G_\alpha \longrightarrow H_{\alpha\eta}$ be a homomorphism such that $\psi_{\alpha\eta,\beta\eta}\chi_\alpha = \chi_\beta\varphi_{\alpha,\beta}$ for any $\alpha \leq \beta$. Then the function χ defined on R by $\chi : a \rightarrow a\chi_\alpha$ if $a \in G_\alpha$, is a homomorphism of R into S. Conversely, every homomorphism R into S can be so constructed.* □

3 Projective semimodules over additively regular semirings with non-empty sets of characters

The results of this section are due to Y.Katsov [4].

Let $\pi : R \longrightarrow S$ be a homomorphism of semirings. Any right S-semimodule X may be considered as a right R-semimodule, denoted $\pi^{\#}X$, by defining $x \cdot r = x\pi(r)$ for any $x \in X,\ r \in R$. One can easily see that the assignments $X \longrightarrow \pi^{\#}X$ are obviously raised to the restriction functor $\pi^{\#} : \mathcal{M}_S \longrightarrow \mathcal{M}_R$. On the other hand, thinking of S as a left R-semimodule ($r \cdot s = \pi(r)s,\ r \in R,\ s \in S$), we have the extension functor $\pi_{\#} \stackrel{def}{=} - \otimes_R S : \mathcal{M}_R \longrightarrow \mathcal{M}_S$. As is shown in [4], $\pi_{\#}$ is a left adjoint to $\pi^{\#}$.

Before proving the main results of this section we state the following four propositions of [4].

Proposition 3.1. *The extension functor $\pi_{\#} : \mathcal{M}_R \to \mathcal{M}_S$ preserves the subcategories of free, projective, finitely generated free and finitely generated projective semimodules, and all colimits.* □

Proposition 3.2. *If $\pi : R \longrightarrow S$ is a surjective semiring homomorphism, then the functors $\pi_{\#}\pi^{\#}$ and $Id_{\mathcal{M}_S} : \mathcal{M}_S \longrightarrow \mathcal{M}_S$ are naturally isomorphic.* □

A semiring $R = (R, +, 0, \cdot, 1)$ is additively regular if $(R, +, 0)$ is a regular monoid. Let R be an additively regular semiring and $[Y; G_\alpha, \varphi_{\alpha,\beta}]$ the Clifford representation of $(R, +, 0)$, *i.e.*, $(R, +, 0) = [Y; G_\alpha, \varphi_{\alpha,\beta}]$. Then G_r will denote the abelian group of this representation that contains the element $r \in R$, and $0_r \in G_r$ the zero (additive identity) of the group G_r.

A semiring R is additively idempotent if $(R, +, 0)$ is an idempotent monoid, *i.e.*, if for any $r \in R$, we have $r + r = r$.

Proposition 3.3. *Let R be an additively regular semiring and $R^0 = \{0_r |\ r \in R\}$. Then, with respect to the operations defined on R, R^0 becomes an additively idempotent semiring with 0_0 and 0_1 as the additive and multiplicative identities, respectively. Also, the multiplication of elements of R by 0_1 produces the surjective semiring homomorphism $0_1 : R \longrightarrow R^0$, and hence, the restriction functor $0_1^{\#} : \mathcal{M}_{R^0} \longrightarrow \mathcal{M}_R$.* □

Proposition 3.4. *The restriction functor $0_1^{\#} : \mathcal{M}_{R^0} \longrightarrow \mathcal{M}_R$ preserves (finitely generated) projective R^0-semimodules.* □

Let $\mathbf{2} = \{0, 1\}$ be the boolean semiring $(1 + 1 = 1)$. A character of a semiring R is a homomorphism of semirings from R to $\mathbf{2}$ [9].

Now we can prove the following theorems (the proofs are taken from [4] without any changes).

Theorem 3.5. *Let R be an additively regular semiring with non-empty set of characters. Then, in the category $\mathcal{M}_R$ of right R-semimodules, the full subcategories of free (finitely generated free) and projective (finitely generated projective) (right) R-semimodules do not coincide. The left-sided analogue of this statement is also valid.*

Proof. First we look at the case when R is an additively idempotent semiring, and suppose, in $\mathcal{M}_R$, the full subcategories of free (finitely generated free) and projective (finitely generated projective) R-semimodules coincide. Since any additively idempotent semiring is obviously additively regular, we may assume that R is an additively regular semiring which is not a ring, and $xR \oplus yR$ (here and below, where context makes it clear, R is thought as a right R-semimodule) is a free two-generated R-semimodule, i.e., $xR \cong R \cong yR$. Then, consider the two R-homomorphisms

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} xR \oplus yR$$

that are defined on the generator $1 \in R$ by $\alpha(1) = (0_1^x, 0_1^y), \beta(1) = (0_x, 0_1^y)$, where $0_x, 0_y$ are zeros of xR and yR, and $0_1^x, 0_1^y$ are zeros of their abelian groups $G_1^x \subset xR$ and $G_1^y \subset yR$, respectively. (Since R is not a ring, clearly $0_1^x \neq 0_x$ and $0_1^y \neq 0_y$.) Now, if τ denotes the $\mathcal{M}_R$-congruence on $xR \oplus yR$ generated by $\langle (0_1^x, 0_1^y), (0_x, 0_1^y) \rangle$, and $\gamma : xR \oplus yR \longrightarrow (xR \oplus yR)/\tau$ its canonical surjection, we obtain the exact sequence

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} xR \oplus yR \xrightarrow{\ \gamma\ } (xR \oplus yR)/\tau \tag{1}$$

in $\mathcal{M}_R$ (meaning that γ is a coequalizer of α and β).

Let $\rho : xR \oplus yR \longrightarrow xR \oplus yR$ be the homomorphism defined on the generators $(1^x, 0_y), (0_x, 1^y)$ of $xR \oplus yR$ by $\rho(1^x, 0_y) = (0_1^x, 0_y)$ and $\rho(0_x, 1^y) = (0_1^x, 0_1^y)$. As ρ is also a homomorphism of additively regular monoids, and $(1^x, 0_y)$ and the idempotent $(0_1^x, 0_y)$ belong to the same abelian group in the monoid $xR \oplus yR$, by Proposition 2.2 $\rho(1^x, 0_y) = \rho(0_1^x, 0_y)$; similarly, $\rho(0_x, 1^y) = \rho(0_x, 0_1^y)$.

At this point of the proof, we will use the additive idempotentness of the semiring R; thus, $1^x = 0_1^x$ and $1^y = 0_1^y$, and, therefore, $\gamma(1^x, 0_y) = \gamma(0_1^x, 0_y)$, and $\gamma(0_x, 1^y) = \gamma(0_x, 0_1^y)$. Then, $\rho\alpha(1) = \rho(0_1^x, 0_1^y) = \rho(0_1^x, 0_y) + \rho(0_x, 0_1^y) = (0_1^x, 0_y) + (0_1^x, 0_1^y) = (0_1^x, 0_1^y)$, and $\rho\beta(1) = \rho(0_x, 0_1^y) = (0_1^x, 0_1^y)$. Hence, there exists $\mu : (xR \oplus yR)/\tau \longrightarrow xR \oplus yR$ such that $\mu\gamma = \rho$ and, therefore, $\gamma\mu\gamma = \gamma\rho$; moreover, since $\gamma(1^x, 0_y) = \gamma(0_1^x, 0_y) = \gamma\rho(1^x, 0_y)$ and $\gamma(0_x, 1^y) = \gamma(0_x, 0_1^y) = \gamma(0_1^x, 0_1^y) = \gamma\rho(0_x, 1^y)$, one has $\gamma\rho = \gamma$. Thus, $\gamma\mu = 1_{(xR\oplus yR)/\tau}$, whence $(xR \oplus yR)/\tau$ is a projective $\mathcal{M}_R$-semimodule, and therefore, according to our assumption, is free.

Then, since there exists a surjective semiring homomorphism $\pi : R \longrightarrow \mathbf{2}$, applying the extension functor $\pi_\# : \mathcal{M}_R \longrightarrow \mathcal{M}_\mathbf{2}$ to the exact sequence (1), by Proposition 3.1, in $\mathcal{M}_\mathbf{2}$ we obtain the exact sequence

$$R \otimes_R \mathbf{2} \underset{\beta\otimes 1}{\overset{\alpha\otimes 1}{\rightrightarrows}} (xR \oplus yR) \otimes_R \mathbf{2} \overset{\gamma\otimes 1}{\longrightarrow} ((xR \oplus yR)/\tau) \otimes_R \mathbf{2}, \tag{2}$$

where the coequalizer $((xR \oplus yR)/\tau) \otimes_R \mathbf{2}$ is a free $\mathcal{M}_\mathbf{2}$-semimodule. Again using Proposition 3.1, one may readily conclude that the exact sequence (2), in fact, can be rewritten as the following exact sequence

$$\mathbf{2} \underset{\beta^*}{\overset{\alpha^*}{\rightrightarrows}} (x\mathbf{2} \oplus y\mathbf{2}) \overset{\gamma^*}{\longrightarrow} (x\mathbf{2} \oplus y\mathbf{2})/\tau^*, \tag{3}$$

where α^* and β^* are completely defined by the maps $1 \longmapsto (x, y)$ and $1 \longmapsto (0, y)$, respectively; τ^* is the congruence on $x\mathbf{2} \oplus y\mathbf{2}$ generated by the pair $\langle (x, y), (0, y) \rangle$, and γ^* the canonical surjection. However, from the latter it is easy to see that $(x\mathbf{2} \oplus y\mathbf{2})/\tau^*$ is isomorphic to the three -element chain $(0, 0) < (x, 0) < (x, y)$ in $x\mathbf{2} \oplus y\mathbf{2}$, which obviously is not a free $\mathbf{2}$-semimodule. Thus, we have established the theorem for an additively idempotent semiring R with non-empty set of characters.

Now let R be an additively regular semiring, and $\pi : R \longrightarrow \mathbf{2}$ a surjective semiring homomorphism. Then, using Proposition 2.2, one can easily see that that the restriction of π on the additively idempotent semiring R^0 gives the surjective homomorphism $\pi|_{R^0} : R^0 \longrightarrow \mathbf{2}$. Therefore, there exists a finitely generated projective (right) R^0-semimodule P that is not free in $\mathcal{M}_{R^0}$. Then, by applying Proposition 3.3 and 3.2, one obtains that $P \cong 0_{1\#} 0_1^\# P$ in $\mathcal{M}_{R^0}$; whence by Propositions 3.4 and 3.1, we conclude that $0_1^\# P \in \mathcal{M}_R$

is a finitely generated projective, but not free, (right) R-semimodule.

The proof of the left-sided analogue of the statement is similar. $\square$

A semiring R is a division semiring if all its nonzero elements are multiplicatively invertible; and a semifiled is a commutative division semiring (see [2]).

Theorem 3.6. *The classes of projective and free right (left) semimodules over the polynomial semiring $R[x_1, x_2, ..., x_n]$ over an additively regular division semiring R coincide iff R is a field.*

Proof. It suffices to show that if R is an additively regular division semiring that is not a ring, then in the category of $R[x_1, x_2, ..., x_n]$-semimodules the category of free (finitely generated free) semimodules is a proper subcategory of the category of projective (finitely generated projective) semimodules.

Thus, let R be an additively regular division semiring that is not a ring. Then, the zeros $0 \in R$ and $0_1 \in G_1 \subset R$ are different; hence, there exists 0_1^{-1} such that $0_1 0_1^{-1} = 0_1^{-1} 0_1 = 1$. However, the multiplication by 0_1^{-1} determines the endomorphism $- \cdot 0_1^{-1} : (R, +) \longrightarrow (R, +)$ of the additive reduct $(R, +)$ of the additively regular semiring R. Therefore by Proposition 2.2, one has, $0_1 0_1^{-1} = 0_1 = 1$. Hence 1 is an idempotent in $(R, +)$, and R is an additively idempotent semiring.

Next, if $a, b \in R \setminus \{0\}$ one may say that $(a + b) \in R \setminus \{0\}$, as well. Indeed, if $a + b = 0$, then $(a + b)a^{-1} = aa^{-1} + ba^{-1} = 1 + ba^{-1} = 0$; whence any element $c \in R$ is additively invertible since $c + cba^{-1} = c(1 + ba^{-1}) = 0$, what contradicts the fact that R is a semiring which is not a ring. From this observation, we conclude that there exists the surjective homomorphism $\chi : R \longrightarrow \mathbf{2}$ that moves $R \setminus \{0\}$ to $1 \in \mathbf{2}$. Therefore, combining the obvious projection $\nu : R[x_1, x_2, ..., x_n] \longrightarrow R$ onto the constant terms with χ, one obtains the surjective homomorphism $\pi : R[x_1, x_2, ..., x_n] \longrightarrow \mathbf{2}$. Now, since $R[x_1, x_2, ..., x_n]$ is clearly an additively regular semiring, by Theorem 3.5, we end the proof.

$\square$

4 Projective semimodules over semirings with valuations in nonnegative integers

The definitions, examples and results (and the proofs) of this section are taken from [5].

The following standard notations are used: if M is a monoid then $U(M)$ is the group of all invertible elements of M; if R is a semiring then $U(R,+,0)$ is the group of all additively invertible elements of R, $U(R)$ the group of all multiplicatively invertible elements of R, and $R^* = R \setminus \{0\}$.

The semiring of all non-negative integers is denoted by $\mathbf{N}$.

Definition 4.1. Let R be a semiring. A function $v : R \longrightarrow \mathbf{N}$ is called a (left) $\mathbf{N}$-valuation of R if the following conditions hold:

(i) $v(r + r') > v(r')$ whenever $r \neq 0$;

(ii) $v(rr') > v(r')$ whenever $r \neq 0$, $r' \neq 0$ and $r \notin U(R)$;

(iii) $v(rr') = v(r')$ for all $r \in U(R)$ and all $r' \in R$.

It immediately follows that $v(r) = 0$ implies $r = 0$, and $v(r) = 1$ implies $r \in U(R)$. At the same time v need not satisfy $v(0) = 0$ or $v(r) = 1$ for $r \in U(R)$ (see 4.3). Also note that if a semiring admits an $\mathbf{N}$-valuation then any multiplicatively left (right) invertible element in it is in fact multiplicatively invertible.

A semiring with an $\mathbf{N}$-valuation is called an $\mathbf{N}$-valued semiring.

Proposition 4.2. *Let R be an $\mathbf{N}$-valued semiring. Then:*

(a) $U(R,+,0) = 0$.

(b) *R is a semiring without zero divisors.*

(c) *If $r + r' = 1$, $r, r' \in R$, then $r = 0$ or $r' = 0$.*

(d) *$R \backslash U(R)$ is a two-sided ideal of R, i.e., R is a local semiring.*

Proof. (a) Suppose $U(R,+,0) \neq 0$. That is, $r + r' = 0$ for some $r, r' \in R^*$. Then $v(0) = v(r + r') > v(r') = v(r' + 0) > v(0)$, a contradiction. Hence $U(R,+,0) = 0$.

(b) Assume $rr' = 0$ for some $r, r' \in R^*$. Then $v(0) = v(rr') > v(r') = v(r' + 0) > v(0)$, a contradiction. Hence R is a zero-divisor-free semiring.

(c) Let $r + r' = 1$ for some $r, r' \in R^*$. Then $v(1) = v(r + r') > v(r') = v(r' \cdot 1) \geq v(1)$, a contradiction. Consequently, $r + r' = 1$, $r, r' \in R$, implies $r = 0$ or $r' = 0$.

(d) It suffices to show that $J := R \backslash U(R)$ is a left ideal.(Indeed, suppose J is a left ideal. Let $\alpha \in J$, $\beta \in R$ and $\alpha\beta \notin J$. Then $\alpha\beta\gamma = 1$ and $\beta\gamma \notin J$ for some $\gamma \in R$, whence $\alpha \in U(R)$, a contradiction. Thus J is a two-sided ideal.) Let $r_1, r_2 \in J$ and $r_1 + r_2 \notin J$. Then there exists $r \in U(R)$ such that $r_1 r + r_2 r = 1$. This gives, by (c), that $r_1 r = 0$ or $r_2 r = 0$. Hence $r_2 \in U(R)$ or $r_1 \in U(R)$, contrary to $r_1, r_2 \in J$. Thus J is a submonoid of the monoid $(R, +, 0)$. Now suppose $\rho \in R$, $\omega \in J$, and $\rho\omega \in U(R)$. Then $v(1) = v(\rho\omega \cdot 1) = v(\rho\omega) > v(\omega) = v(\omega \cdot 1) > v(1)$, a contradiction. Thus J is a left ideal. □

Example 4.3. $\mathbf{N}$ is evidently $\mathbf{N}$-valued: $v = 1 : \mathbf{N} \longrightarrow \mathbf{N}$. Furthermore, a function $v : \mathbf{N} \longrightarrow \mathbf{N}$ is an $\mathbf{N}$-valuation if and only if it is a strictly increasing function.

Example 4.4. Let $\mathbf{R}$ be the field of real numbers. Then $\{0, 1\} \cup \{r \in \mathbf{R} \,|\, r \geq 2\}$ is a subsemiring of $\mathbf{R}$, and the greatest integer function $[\,] : \{0, 1\} \cup \{r \in \mathbf{R} \,|\, r \geq 2\} \longrightarrow \mathbf{N}$ is an $\mathbf{N}$-valuation.

Example 4.5. For any semiring R,

$$R' = \big\{(r, n) \in R \times \mathbf{N} \,|\, n \geq 2\big\} \cup \big\{(r, 1) \in R \times \mathbf{N} \,|\, r \in U(R)\big\} \cup \big\{(0, 0)\big\}$$

is an $\mathbf{N}$-valued subsemiring of $R \times \mathbf{N}$. Indeed, $v : R' \longrightarrow \mathbf{N}$, $v(r, n) = n$, is an $\mathbf{N}$-valuation.

This and many more examples of $\mathbf{N}$-valued semirings can be drawn from

Proposition 4.6. *Let R be an $\mathbf{N}$-valued semiring and $\varphi : R' \longrightarrow R$ a homomorphism of semirings such that* $\ker(\varphi) := \{r' \in R' | \varphi(r') = 0\} = 0$ *and* $\varphi^{-1}(U(R)) = U(R')$. *Then R' is an $\mathbf{N}$-valued semiring.*

Proof. If $v : R \longrightarrow \mathbf{N}$ is an $\mathbf{N}$-valuation, then so is $v\varphi : R' \longrightarrow \mathbf{N}$. □

Note that rings as well as semifields do not admit any $\mathbf{N}$-valuations (see 4.2(a) and 4.2(c)).

Definition 4.7. We say that a monoid M is $\mathbf{N}$-valued if there exists a function $p : M \longrightarrow \mathbf{N}$, called a (left) $\mathbf{N}$-valuation of M, such that

(i) $p(xy) > p(y)$ for all $x \in M \backslash U(M)$ and all $y \in M$;

(ii) $p(xy) = p(y)$ for all $x \in U(M)$ and all $y \in M$.

Example 4.8. Any group G is evidently **N**-valued: $p(g) = 0$ for all $g \in G$. Further, let $F(T)$ be a free monoid on a set T. The function $\deg : F(T) \longrightarrow \mathbf{N}$ assigning to $x \in F(T)$ its degree (note that any free monoid has a unique basis) is an **N**-valuation. Clearly, the restriction of deg to any submonoid of $F(T)$ is also an **N**-valuation. Analogously, free abelian monoids and their submonoids are **N**-valued monoids.

If $\psi : M' \longrightarrow M$ is a homomorphism of monoids with $\psi^{-1}(U(M)) = U(M')$ and M is **N**-valued, then M' is **N**-valued (cf. 4.6). This together with 4.8 enable us to obtain more examples of **N**-valued monoids.

Proposition 4.9. *Let R be a semiring and M a monoid. If R and M are both* **N***-valued, then so is the monoid semiring $R[M]$.* □

From 4.8 and 4.9 we get the following corollaries.

Corollary 4.10. *Let R be an* **N***-valued semiring and G an arbitrary group. Then $R[G]$ is an* **N***-valued semiring.* □

Corollary 4.11. *Let R be an* **N***-valued semiring and E be either a submonoid of a free monoid, or a submonoid of a free abelian monoid. Then $R[E]$ is an* **N***-valued semiring.* □

4.12. Let R be a semiring with an **N**-valuation $v : R \longrightarrow \mathbf{N}$, and let $F(T)$ be a free R-semimodule on a set T. Then $l : F(T) \longrightarrow \mathbf{N}$ defined by

$$l\Big(\sum_{t\in T} r_t t\Big) = v\Big(\sum_{t\in T} r_t\Big)$$

satisfies the following conditions:

(i) $l(a + b) > l(b)$ for all $a \in F(T)\backslash\{0\}$ and $b \in F(T)$;

(ii) $l(rb) > l(b)$ for all $r \in R\backslash(U(R) \cup \{0\})$ and $b \in F(T)\backslash\{0\}$;

(iii) $l(rb) = l(b)$ for all $r \in U(R)$ and $b \in F(T)$.

(One may say that l is an **N**-valuation of the R-semimodule $F(T)$.) As an immediate consequence of (i), (ii) and (iii) we have:

(iv) If $a = r_1a_1 + \cdots + r_ma_m$, $r_1, \ldots, r_m \in R$, $a_1, \ldots, a_m \in F(T)$, $m > 1$, and $r_1a_1 \neq 0, \ldots, r_ma_m \neq 0$, then $l(a) > l(a_j)$, $j = 1, \ldots, m$.

Now we are ready to prove the main result and state some of its corollaries.

Theorem 4.13. *If R is an $\mathbf{N}$-valued semiring, then any projective R-semimodule is free.*

Proof. Let P be a non-trivial projective R-semimodule. Since any projective R-semimodule is a retract of a free R-semimodule, there is a diagram

$$F(T) \underset{j}{\overset{\pi}{\rightleftarrows}} P \ ,$$

where $F(T)$ is a free R-semimodule on a set T, π and j are R-homomorphisms, and $\pi j = 1$. Clearly, in addition, one may assume that j is an inclusion and that $\pi(t) \neq 0$ for any $t \in T$. We show that the set

$$S = \big\{ s \in T \,|\, s = r\pi(t) \ \text{ for some } \ t \in T \ \text{ and } \ r \in U(R) \big\}$$

is an R-basis of P. Obviously, since $S \subset T$ and $\pi(T)$ is a set of R-generators for the R-semimodule P, it suffices to see that for any $t \in T$, $\pi(t) = \sum_{s\in S} r^t_s s$, $r^t_s \in R$.

As R is an $\mathbf{N}$-valued semiring, there is, as noted above, a function $l : F(T) \longrightarrow \mathbf{N}$ satisfying 4.12(i)–(iv). The function l and the set $\pi(T)$ uniquely determine a strictly increasing (finite or infinite) sequence

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

of positive integers as follows. A positive integer n is a term of this sequence if and only if there exists $t \in T$ with $l(\pi(t)) = n$ (in view of 4.12(i), $l(a) > 0$ whenever $a \neq 0$). Next, for any $t \in T$, one has the representation of $\pi(t)$ by the R-basis T:

$$\pi(t) = r^t_1 t_1 + \cdots + r^t_m t_m, \ \ r^t_1, \ldots, r^t_m \in R^*, \ \ t_1, \ldots, t_m \in T. \qquad (*)$$

Applying π to $(*)$ and using 2.1 and 4.2(b), we obtain

$$\pi(t) = r^t_1 \pi(t_1) + \cdots + r^t_m \pi(t_m), \ \ r^t_1 \pi(t_1) \neq 0, \ldots, r^t_m \pi(t_m) \neq 0. \qquad (**)$$

Let $(*)$ be the representation of $\pi(t)$ with $l(\pi(t)) = n_1$. Then $m = 1$ and $r^t_1 \in U(R)$. Indeed, when $m > 1$ or $r^t_1 \notin U(R)$, we get, by $(**)$ and 4.12(ii),(iv), that $n_1 = l(\pi(t)) > l(\pi(t_1))$, contradicting $n_1 = \min\{l(\pi(t)) \,|\, t \in T\}$. Thus, if $l(\pi(t)) = n_1$ then $\pi(t) = rs$, $r \in U(R)$, $s \in S$. This suggests to continue the proof by induction on k. Assume that for any $\tau \in T$ with $l(\pi(\tau)) \leq n_k$ one has $\pi(\tau) = \sum_{s\in S} r_s^\tau s$, and let $(*)$ be the representation of $\pi(t)$ with $l(\pi(t)) = n_{k+1}$. If $m = 1$ and $r_1^t \in U(R)$, then $\pi(t) = rs$, where $r = r_1^t$ and $s = t_1 \in S$. Suppose $m > 1$ or $r_1^t \notin U(R)$. It then follows from $(**)$ and 4.12(ii),(iv) that $l(\pi(t)) > l(\pi(t_j))$, $j = 1, \ldots, m$. That is, $l(\pi(t_j)) \leq n_k$, $j = 1, \ldots, m$. Hence, by the induction assumption, $\pi(t_j) = \sum_{s\in S} r_s^{(j)} s$, $j = 1, \ldots, m$. Consequently, since $\pi(t) = \sum_{j=1}^{m} r_j^t \pi(t_j)$, one has $\pi(t) = \sum_{s\in S} \big(\sum_{j=1}^{m} r_j^t r_s^{(j)} \big) s$. □

Corollary 4.14 ([3])**.** *Any projective* **N***-semimodule (i.e., any projective abelian monoid) is free.* □

Proposition 4.9 and Theorem 4.13 yield

Corollary 4.15. *If R is an* **N***-valued semiring and M an* **N***-valued monoid, then any projective $R[M]$-semimodule is free.* □

This theorem together with Corollary 4.10 gives

Corollary 4.16. *Let R be an* **N***-valued semiring and G an arbitrary group. Then any projective $R[G]$-semimodule is free.* □

In particular, we have

Corollary 4.17. *For any group G, all projective* **N***$[G]$-semimodules are free.* □

Theorem 4.13 and Corollary 4.11 give

Corollary 4.18. *Let R be an* **N***-valued semiring and suppose that E is either a submonoid of a free monoid, or a submonoid of a free abelian monoid. Then any projective $R[E]$-semimodule is free.* □

As a special case of 4.18 we single out

Corollary 4.19. *For any* **N***-valued semiring R the classes of projective and free semimodules over the polynomial semiring $R[x_1, \ldots, x_n]$ coincide. In particular, all projective* **N***$[x_1, \ldots, x_n]$-semimodules are free.* □

5 The Grothendieck Group of an N-Valued Semiring

In this section, using Theorem 4.13, we calculate the Grothendieck group K_0R of an **N**-valued Semiring R.

5.1. Let R be a semiring without zero divisors and with $U(R, +, 0) = 0$. If $r_1, \dots, r_n, r'_1, \dots, r'_n$ are nonzero elements of R, then $r_1r'_1 + \cdots + r_nr'_n \neq 0$.

Proposition 5.2. *Let R be a semiring without zero divisors and with $U(R, +, 0) = 0$, and suppose that 1 is additively irreducible, i.e., whenever $r + r' = 1$, $r, r' \in R$, one has $r = 0$ or $r' = 0$. Suppose further that F is a free R-semimodule and $S, T \subset F$ are R-bases of F. Then for any $s \in S$ there exists a unique $r_s \in U(R)$ such that $r_s s \in T$.*

Proof. Let $s_0 \in S$. Represent s_0 by the R-basis T:

$$s_0 = r_1t_1 + \cdots + r_nt_n, \quad r_1 \neq 0, \dots, r_n \neq 0.$$

On the other hand, for each i, $i = 1, \dots, n$, one has the representation of t_i by the R-basis S:

$$t_1 = \sum_{s\in S} r_s^{(1)} s, \dots, t_n = \sum_{s\in S} r_s^{(n)} s.$$

So we have

$$s_0 = \sum_{s\in S}(r_1r_s^{(1)} + \cdots + r_nr_s^{(n)})s,$$

whence

$$r_1r_{s_0}^{(1)} + \cdots + r_nr_{s_0}^{(n)} = 1, \text{ and } r_1r_s^{(1)} + \cdots + r_nr_s^{(n)} = 0 \text{ for all } s \neq s_0.$$

From the latter we conclude that

$$r_s^{(1)} = 0, \dots, r_s^{(n)} = 0 \text{ for all } s \in S\backslash\{s_0\}$$

since $r_1 \neq 0, \dots, r_n \neq 0$, $U(R, +, 0) = 0$ and R is a zero-divisor-free semiring. Consequently,

$$t_1 = r_{s_0}^{(1)} s_0, \dots, t_n = r_{s_0}^{(n)} s_0,$$

whence $r_{s_0}^{(1)} \neq 0, \dots, r_{s_0}^{(n)} \neq 0$.

As noted above $r_1r_{s_0}^{(1)} + \cdots + r_nr_{s_0}^{(n)} = 1$. Suppose $n > 1$. Then, by 5.1, $r_1r_{s_0}^{(1)} \neq 0$ and $r_2r_{s_0}^{(2)} + \cdots + r_nr_{s_0}^{(n)} \neq 0$, a contradiction to our assumption that 1 is additively irreducible. Hence $n = 1$. So, in fact, $s_0 = r_1t_1$. This together with $t_1 = r_{s_0}^{(1)}s_0$ gives $s_0 = r_1r_{s_0}^{(1)}s_0$ and $t_1 = r_{s_0}^{(1)}r_1t_1$, whence $r_1r_{s_0}^{(1)} = 1$ and $r_{s_0}^{(1)}r_1 = 1$. Thus we have $r_{s_0}^{(1)}s_0 = t_1 \in T$ and $r_{s_0}^{(1)} \in U(R)$. As T is an R-basis of F, the uniqueness of r_s is obvious. □

Corollary 5.3. *Let $R, F, S,$ and T be as in* 5.2. *Then* $\mathrm{card}(S) = \mathrm{card}(T)$.

Proof. For any $s \in S$ there exists, by 5.2, a unique $r_s \in U(R)$ with $r_ss \in T$. Define $\theta : S \longrightarrow T$ by $\theta(s) = r_ss$. Since S is an R-basis of F, θ is one-to-one. Let $t \in T$. By 5.2, there exists a unique $r_t \in U(R)$ such that $r_tt = s \in S$. Clearly, $r_s = r_t^{-1}$. Hence $\theta(s) = t$. Thus θ is onto. □

Let R be a semiring. Recall [8] the construction of K_0R. Let $\mathbf{P}(R)$ denote the class of all finitely generated projective R-semimodules and let $\langle P\rangle$ denote the isomorphism class of $P \in \mathbf{P}(R)$. K_0R is the abelian group with generators $\langle P\rangle$, $P \in \mathbf{P}(R)$, and relations $\langle P_1\rangle + \langle P_2\rangle = \langle P_1 \oplus P_2\rangle$, $P_1, P_2 \in \mathbf{P}(R)$.

It was shown in [8] that $K_0\mathbf{N}$ is the infinite cyclic group generated by class$\langle \mathbf{N}\rangle$. The following statement generalizes this result to $\mathbf{N}$-valued semirings.

Proposition 5.4. *For any $\mathbf{N}$-valued semiring R, K_0R is the infinite cyclic group generated by* class$\langle R\rangle$.

Proof. Let $k, k' \in \mathbf{N}$. By 5.3 and 4.2, R^k is isomorphic to $R^{k'}$ if and only if $k = k'$. That is, $\langle R^k\rangle = \langle R^{k'}\rangle$ iff $k = k'$. From this and Theorem 4.13 we have $K_0R = F/H$, where F is the free abelian group generated by $\langle R\rangle, \langle R^2\rangle, \ldots, \langle R^n\rangle, \ldots,$ and H the subgroup of F generated by all elements of the form $\langle R^m\rangle + \langle R^n\rangle - \langle R^{m+n}\rangle$, $m, n > 0$. Clearly, $k \cdot \mathrm{class}\langle R\rangle = \mathrm{class}\langle R^k\rangle$, $k \in N$. Hence K_0R is a cyclic group generated by class$\langle R\rangle$. It then remains to prove that $k \cdot \mathrm{class}\langle R\rangle = 0$ implies $k = 0$. Let Z be the additive group of integers and $\theta : F \longrightarrow Z$ the homomorphism defined by $\theta(\langle R^k\rangle) = k$. Assume that $k \cdot \mathrm{class}\langle R\rangle = 0$, i.e., $\mathrm{class}\langle R^k\rangle = 0$. Then

$$\langle R^k\rangle = \sum_i a_i\big(\langle R^{m_i}\rangle + \langle R^{n_i}\rangle - \langle R^{m_i+n_i}\rangle\big), \quad a_i \in Z.$$

Applying θ to this, we get $k = 0$. □

6 Irreducible elements

In this section we show that Theorem 4.13 can be proved using only (i) and (ii) of 4.1.

Definition 6.1. *A non-zero element of an R-semimodule A is said to be R-irreducible if the following conditions are satisfied:*
(i) u is additively irreducible, that is, whenever $u = a+b$, $a, b \in A$, one has $a = 0$ or $b = 0$.
(ii) whenever $u = ra$, $a \in A$, $r \in R$, one has $r \in U(R)$.

The set of all R-irreducible elements of an R-semimodule A will be denoted by $I_R(A)$.

Proposition 6.2. *let R be a semiring and $v : R \to N$ a function satisfying (i) and (ii) of 4.1. Suppose further that $F(T)$ is a free R-semimodule on a set T. Then any R-subsemimodule A of $F(T)$ is generated by $I_R(A)$.*

Proof. Define $l : A \to N$ by

$$l(\sum_{t \in T} r_t t) = v(\sum_{t \in T} r_t).$$

Clearly,

(l1) $l(a + b) > l(b)$ for all $a \in A \setminus \{0\}$ and $b \in A$.

and

(l2) $l(ra) > l(a)$ for all $r \in R \setminus (U(R) \cup \{0\})$ and $a \in A \setminus 0$.

(Note that $ra = 0$ implies $r = 0$ or $a = 0$.) The function $l : A \to N$ uniquely determines a strictly increasing sequence

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

of positive integers as follows. A positive integer n is a term of this sequence if and only if there exists $a \in A$, $a \neq 0$, with $l(a) = n$. It immediately follows from (l1) and (l2) that any $a \in A$ with $l(a) = n_1$ is R-irreducible in A. Assume now that any $x \in A$ with $l(x) \leq n_k$ has a representation of the form

$$x = \sum_{u \in I_R(A)} r_u u, \; r_u \in R,$$

and take any $a \in A$ with $l(a) = n_{k+1}$. If $a \in I_R(A)$, there is nothing to prove. Suppose that $a \notin I_R(A)$. If 6.1(i) does not hold, then $a = b + c$, $b, c \in A$, $b \neq 0$ $a \neq 0$. By (l1),

$l(b) < l(a)$ and $l(c) < l(a)$. That is, $l(b) \leq n_k$ and $l(c) \leq n_k$. Therefore by the induction assumption,

$$b = \sum_{u \in I_R(A)} r'_u u, \;\; c = \sum_{u \in I_R(A)} r''_u u, \; r'_u, r''_u \in R$$

whence

$$a = \sum_{u \in I_R(A)} (r'_u + r''_u) u.$$

If 6.1 (ii) does not hold for a, then $a = rd$, $d \in A$, $r \in R \setminus U(R)$. This by (12), gives $l(d) < l(a)$. That is $l(d) \leq n_k$. Consequently, by the induction assumption,

$$d = \sum_{u \in I_R(A)} r_u u, \; r_u \in R,$$

whence

$$a = \sum_{u \in I_R(A)} (r r_u) u$$

.

□

Proposition 6.3. *Let R be a semiring with $U(R, +, 0) = 0$ and without zero devisors and let P be a projective R-semimodule. If P is generated by $I_R(P)$, then P is a free R-semimodule.*

Proof. There is a diagram

$$F(T) \underset{j}{\overset{\pi}{\rightleftarrows}} P \;\; ,$$

where $F(T)$ is a free R-semimodule on a set T, π and j are R-homomorphisms, and $\pi j = 1$. In addition, one may assume that j is an inclusion and that $\pi(t) \neq 0$ for any $t \in T$. Take any $u \in I_R(P)$ and represent it by the R-basis T:

$$u = r_1 t_1 + \cdots + r_n t_n, \; t_1, \cdots, t_n \in T, \; r_1, \cdots, r_n \in R \setminus \{0\}$$

whence

$$u = r_1 \pi(t_1) + \cdots + r_n \pi(t_n).$$

This implies $n = 1$ and $r_1 \in U(R)$ since R is a zero-devisor-free semiring, $U(R, +, 0) = 0$ and $u \in I_R(P)$. Hence $u = r_1 t_1$, i.e., $t_1 = r_1^{-1} u \in P$. Consequently, P is a free R-semimodule over the set

$$\{t \in T \mid t = ru \text{ for some } u \in I_R(P) \text{ and } r \in U(R)\}$$

□

Propositions 6.2 and 6.3 give

Corollary 6.4. *Let R be a semiring and $v : R \to N$ a function satisfying (i) and (ii) of 4.1. Then any projective R-semimodule is free.*

Proof. It suffices to note that $U(R, +, 0) = 0$ and R is a zero-devisor-free semiring (see the proofs of 4.2(a) and 4.2(b), respectively).

□

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