Universal Coefficient Theorems in KK-theory

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Interactions between C*-algebraic KK-theory and homotopy theory

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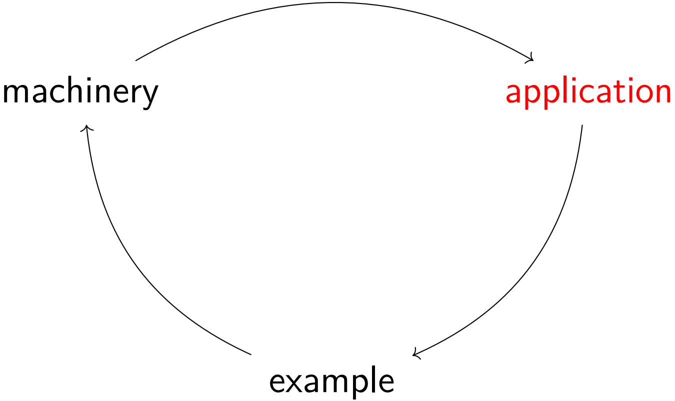
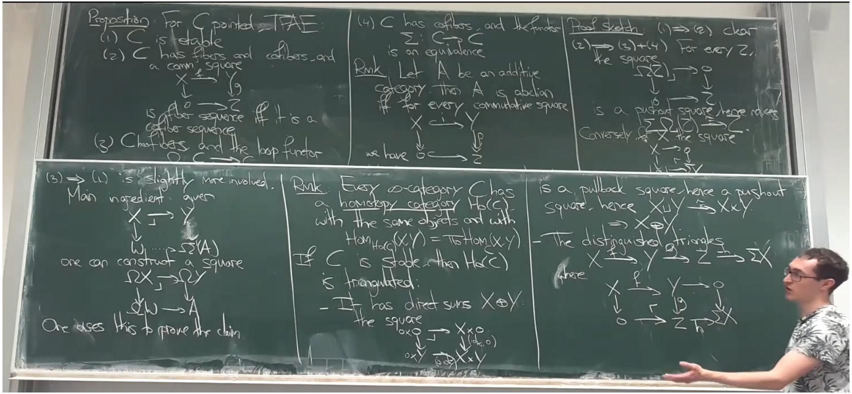

machinery
application
example

Ralf Meyer, George Nadareishvili (2024).
A universal coefficient theorem for actions of finite groups on C*-algebras.
Preprint on arXiv.

Machinery: a triangulated category

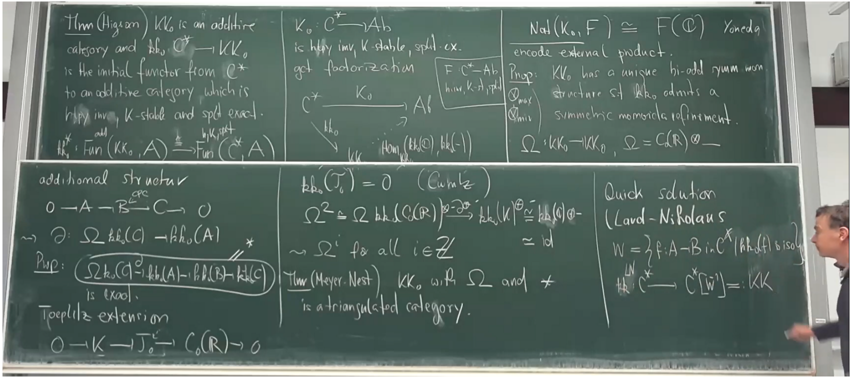


Bastiaan Cnossen: Introduction to stable ∞-categories.

Proposition

Homotopy category of a stable ∞-category is triangulated.

Application: triangulated category $\mathfrak{KK}$



Ulrich Bunke: KK-theory from the point of view of homotopy theory.

Theorem (Meyer–Nest)

$\mathfrak{KK}$ (or $\mathfrak{KK}_0$) with Ω (or Σ) and exact triangles explained is a triangulated category.

Application: triangulated category $\mathfrak{KK}^G$

Equivariant Kasparov theory

Let G be a locally compact group.

Equivariant Kasparov theory defines an additive category KK^G, with

- objects all separable G-C^*-algebras
- morphism sets the bivariant Kasparov K-groups $KK^G(A, B)$
- the composition of morphisms

$$KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C)$$

given by *Kasparov product*.

Christian Voigt: The Baum-Connes conjecture and quantum groups.

Application: triangulated category $\mathfrak{KK}^G$

Structure as a triangulated category

The category KK^G is triangulated - this allows one to do homological algebra.

A triangulated category is an additive category togther with a translation functor and a class of exact triangles satisfying certain axioms.

In the case of KK^G, we have that

- the (inverse of the) *suspension* $\Sigma A = C_0(\mathbb{R}) \otimes A$ yields the translation functor.
- the exact triangles are all diagrams in KK^G isomorphic to mapping cone triangles

$$\Sigma B \to C_f \to A \to B$$

  for equivariant $*$-homomorphisms $f : A \to B$.

Every extension $0 \to I \to A \to B \to 0$ of G-C^*-algebras with a G-equivariant completely positive contractive linear splitting defines an exact triangle.

Christian Voigt: The Baum-Connes conjecture and quantum groups.

Machinery: homological algebra

Systematic organization of computation methods to study structures through notion of exactness.

Triangulated category $\mathfrak{T}$.

- Exact triangles: $A \to B \to C \to \Sigma A$

Aim

Do homological algebra on a triangulated category $\mathfrak{T}$.

- Maybe calculate $\mathfrak{T}(A, B)$?

Procedure

Exactness $\Longrightarrow$ Projective objects $\Longrightarrow$ Projective resolutions $\Longrightarrow$ Derived functors $\Longrightarrow \cdots$

Machinery: exactness

The obvious homological algebra structure in $\mathfrak{T}$ is trivial.

Observation

Non-abelian $\Longrightarrow$ need additional data to get started.

Pick a homological functor into an abelian category (everything is stable)

$$H\colon \mathfrak{T} \to \mathfrak{A}.$$

- Call a triangle $A \to B \to C \to \Sigma A$ an *H-exact* triangle iff
$$0 \to H(A) \to H(B) \to H(C) \to 0$$
is short exact.
- Call a chain complex $C_\bullet$ over $\mathfrak{T}$ an *H-exact* chain complex iff $H(C_\bullet)$ is long exact.
- Call a homological functor $F\colon \mathfrak{T} \to \mathfrak{B}$ an *H-exact functor* iff it maps H-exact triangles to short exact sequences. Motto: what is invisible to H, is invisible to F.

Projective objects and derived functors

- P is called *H-projective* if the functor
$$\mathfrak{T}(P, \sqcup): \mathfrak{T} \to \mathfrak{Ab}$$
is H-exact.
- An *H-projective resolution* of $A \in \mathfrak{T}$ is an H-exact chain complex
$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$$

If $\mathfrak{T}$ has enough projective objects, construction of projective resolutions provides a functor $P: \mathfrak{T} \to \mathfrak{ho}(\mathfrak{T})$.

Definition

Let $F: \mathfrak{T} \to \mathfrak{A}$ be an additive functor into Abelian $\mathfrak{A}$. Define the nth left derived functor of F as
$$\mathbb{L}_n F: \mathfrak{T} \xrightarrow{P} \mathfrak{Ho}(\mathfrak{T}) \xrightarrow{\mathfrak{Ho}(F)} \mathfrak{Ho}(\mathfrak{B}) \xrightarrow{H_n} \mathfrak{A}.$$

Universal Abelian approximation

An H-exact stable homological functor $U\colon \mathfrak{T} \to \mathfrak{A}_U$ is called universal if any other H-exact homological functor G factors through a unique exact functor $\overline{G}$

$$\begin{array}{ccc} \mathfrak{T} & \xrightarrow{\quad U \quad} & \mathfrak{A}_U \\ & {}_{G}\searrow \quad \swarrow_{\overline{G}} & \\ & \mathfrak{B} & \end{array}$$

Homology theory on $\mathfrak{A}_U$ and H-homology theory on $\mathfrak{T}$ are "identified" through U. For example,

$$\mathfrak{Proj}_{\mathfrak{T},H} \xrightarrow[U]{\cong} \mathfrak{Proj}_{\mathfrak{A}_U}$$

and, for any homological $F\colon \mathfrak{T} \to \mathfrak{B}$, there is $\overline{F}\colon \mathfrak{A}_U \to \mathfrak{B}$, with

$$\mathbb{L}_n F(A) \cong \mathbb{L}_n \overline{F} \circ U(A).$$

Specifically,

$$\operatorname{Ext}^n_{\mathfrak{T},H}(A,B) \cong \operatorname{Ext}^n_{\mathfrak{A}_U}(U(A),U(B)).$$

Specific universal approximation

Assumption

From now on, assume $\mathfrak{T}$ has countable coproducts.

- Fix at most countable set $\mathcal{C}$ of compact objects in $\mathfrak{T}$.
- Let $\mathfrak{C}$ denote the small category with $\mathcal{C}$ as its objects and groups of arrows $\mathfrak{C}(C, C') := \mathfrak{T}_*(C, C') = \bigoplus_{n\in\mathbb{Z}} \mathfrak{T}(\Sigma^n C, C')$.

The Yoneda functor into the category of (stable) contravariant additive countable functors

$$\mathfrak{T} \xrightarrow{U_{\mathfrak{C}}} \mathrm{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}})_{\text{countable}}, \qquad A \mapsto (\mathfrak{T}_*(C, A))_{c\in\mathfrak{C}}$$

is the universal $U_{\mathfrak{C}}$-exact homological functor.

Universal Coefficient Theorem

Theorem

Let $A \in \langle \mathfrak{C} \rangle$ (=: Bootstrap class) and $B \in \mathfrak{T}$. There is a natural, cohomologically indexed, right half-plane, conditionally convergent spectral sequence of the form

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{C}}(U_{\mathfrak{C}}(A), U_{\mathfrak{C}}(B))_{-q} \Rightarrow \mathfrak{T}(\Sigma^{p+q} A, B).$$

If the object $U_{\mathfrak{C}}(A)$ has a projective resolution of length 1, then there is a natural short exact sequence

$$\mathrm{Ext}^1_{\mathfrak{C}}(U_{\mathfrak{C}}(\Sigma A), U_{\mathfrak{C}}(B)) \rightarrowtail \mathfrak{T}(A, B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}}(U_{\mathfrak{C}}(A), U_{\mathfrak{C}}(B)).$$

Game

- Start with some class of objects $\mathfrak{B} \subseteq \mathfrak{T}$ we wish to study.
- Pick $\mathfrak{C}$ with nice enough homological properties and $\langle \mathfrak{C} \rangle \cong \mathfrak{B}$.

Universal Coefficient Theorem in $\mathfrak{KK}$

- Pick $\mathfrak{C} = \{\mathbb{C}\}$.
- The universal invariant

$$\mathfrak{T} \to \mathrm{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}})_{\mathsf{countable}}, \qquad A \mapsto (\mathfrak{T}_*(C, A))_{c \in \mathfrak{C}}$$

becomes

$$\mathfrak{KK} \to \mathrm{Fun}(\{\mathbb{C}\}, \mathfrak{Ab}^{\mathbb{Z}})_{\mathsf{c}} \cong \mathfrak{Ab}_{\mathsf{c}}^{\mathbb{Z}/2}, \qquad A \mapsto (\mathfrak{KK}(\mathbb{C}, A), \mathfrak{KK}(\Sigma\mathbb{C}, A)) \cong \mathrm{K}_*(A).$$

As abelian groups have length 1 projective resolutions,

Theorem (Rosenberg-Schochet)

Let A be a separable C^*-algebra. Then for $A \in \langle\mathbb{C}\rangle$, there is a short exact sequence of $\mathbb{Z}/2$-graded abelian groups

$$\mathrm{Ext}^1(\mathrm{K}_*(\Sigma A), \mathrm{K}_*(B)) \rightarrowtail \mathfrak{KK}_*(A, B) \twoheadrightarrow \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$$

for every $B \in \mathfrak{KK}$.

Application: $\mathfrak{KK}^G$ for finite G

- "The equivariant Bootstrap class $\mathfrak{B}^G \subset \mathfrak{KK}^G$" := "Actions on Type I C*-algebras".
- Known that $\mathfrak{B}^G \cong \langle \mathrm{ind}_H^G \mathbb{M}_n(\mathbb{C}) \mid$ all actions for subgroups $H \subseteq G \rangle$.

Note:

$$\mathrm{ind}_H^G \mathbb{C} \cong \mathrm{C}(G/H) \in \mathfrak{B}^G.$$

Theorem (Arano–Kubota 2018, Meyer–N. 2024)

The objects $\mathrm{C}(G/H)$ for cyclic subgroups $H \subseteq G$ already generate the equivariant bootstrap class $\mathfrak{B}^G$ in $\mathfrak{KK}^G$.

Application: $\mathfrak{KK}^G$ for finite G

For $\mathfrak{C}$, we pick

$$\mathfrak{Cyc} = \{\mathsf{C}(G/H) \mid H \text{ is a cyclic subgroup of } G\}.$$

The universal invariant

$$\mathfrak{T} \to \mathrm{Fun}(\mathfrak{C}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}})_{\mathsf{c}}, \qquad A \mapsto (\mathfrak{T}_*(C, A))_{c \in \mathfrak{C}}$$

becomes

$$\mathfrak{KK}^G \xrightarrow{\mathrm{ck}_*^G} \mathrm{Fun}(\mathfrak{Cyc}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}/2})_{\mathsf{c}},$$

where ck_*^G maps

$$A \mapsto \{\mathfrak{KK}_*^G(\mathsf{C}(G/H), A)\}^{\text{cyclic } H \subseteq G} = \{\mathrm{K}_*^H(A)\}^{\text{cyclic } H \subseteq G}.$$

Theorem (Dell'Ambrogio 2014)

The functor $A \mapsto \{\mathrm{K}_*^H(A)\}^{H \subseteq G}$ is a Mackey functor into the category of Mackey modules over a representation Green ring of G.

To compute $\mathfrak{Cyc}$ and $\mathrm{Fun}(\mathfrak{Cyc}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}/2})_{\mathsf{c}}$, we restrict this result to cyclic subgroups.

Example: $V = \mathbb{Z}/2 \times \mathbb{Z}/2$

Klein four-group $\mathbb{Z}/2 \times \mathbb{Z}/2$ has four cyclic subgroups

$$\langle(0,0)\rangle \cong \{0\}, \quad \langle(1,0)\rangle \cong \mathbb{Z}/2, \quad \langle(1,1)\rangle \cong \mathbb{Z}/2, \text{ and } \langle(0,1)\rangle \cong \mathbb{Z}/2.$$

Thus,

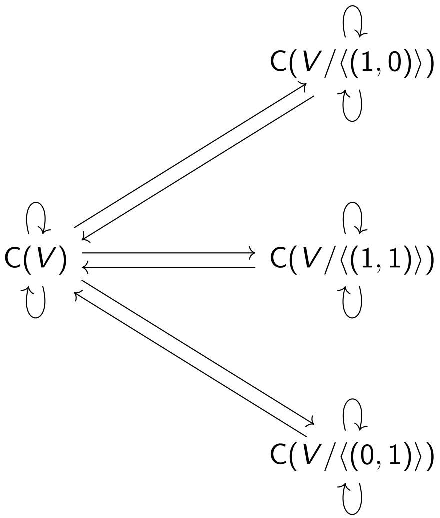
$$\mathrm{Ob}(\mathfrak{C}\mathrm{yc}) = \{\mathsf{C}(V),\ \mathsf{C}(V/\langle(1,0)\rangle),\ \mathsf{C}(V/\langle(1,1)\rangle),\ \mathsf{C}(V/\langle(0,1)\rangle)\}.$$

Generators

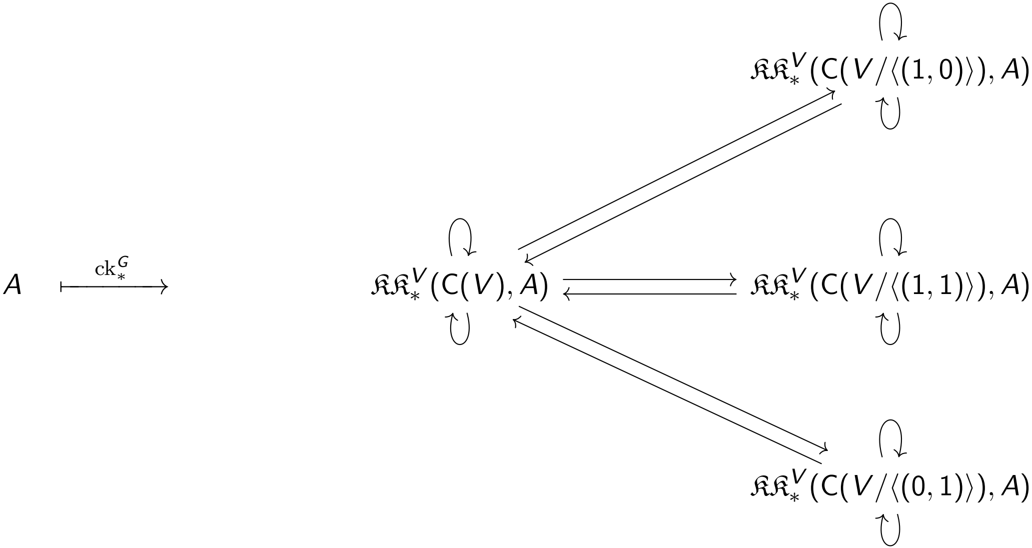
$$\mathsf{C}(V/V) \cong \mathbb{C} \text{ and } \mathrm{ind}_V^V \mathbb{M}_2(\mathbb{C})$$

are redundant.

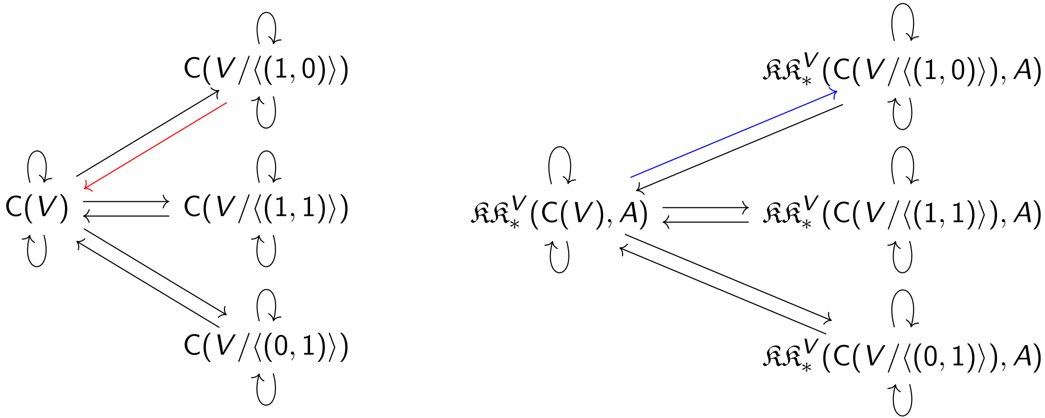
Category $\mathfrak{Cyc}$ for $\mathbb{Z}/2 \times \mathbb{Z}/2$



Universal invariant for $\mathbb{Z}/2 \times \mathbb{Z}/2$



Action of $\mathfrak{C}\mathrm{yc}$ for $\mathbb{Z}/2 \times \mathbb{Z}/2$



Application: $\mathfrak{KK}^G$ for finite G.

Theorem (Meyer–N. 2024)

Let A be an action on a type I C-algebra and $B \in \mathfrak{KK}^G$. There is a natural, cohomologically indexed, right half-plane, conditionally convergent spectral sequence of the form*

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{C}\mathrm{yc}}(\mathrm{ck}^G_*(A), \mathrm{ck}^G_*(B))_{-q} \Rightarrow \mathfrak{KK}^G(\Sigma^{p+q} A, B).$$

Question

Can we get a short exact sequence? That is, a Universal Coefficient Theorem?

Towards the UCT: inverting the group order

Experience

Representation theory of groups becomes relatively easy over a field in which the group order is invertible.

Adopting ideas of Manuel Köhler,

Lemma (Meyer–N. 2024)

For finite G, any object in

$$\mathfrak{A}[|G|^{-1}] := \mathrm{Fun}(\mathfrak{Cyc}^{\mathrm{op}}, \mathfrak{Ab}^{\mathbb{Z}/2})_c[|G|^{-1}]$$

has projective resolution of length 1. We can compute this localisation explicitly.

Question

What class in $\mathfrak{KK}^G$ does $\mathfrak{A}[|G|^{-1}]$ approximate universally? We will have a UCT for that class!

Towards the UCT: group-order-divisible objects

Definition

Let S be a set of primes. A separable G-C*-algebra A is S-divisible if $p \cdot \mathrm{id}_A \in \mathfrak{KK}^G(A, A)$ is invertible for all $p \in S$.

Denote by $\mathbb{M}_{S^\infty}$ the UHF algebra of type $\prod_{p \in S} p^\infty$ with the trivial action of G

Proposition (Meyer–N. 2024)

The S-divisible objects form a (localising) subcategory $\mathfrak{KK}^G_S \subset \mathfrak{KK}^G$. The class

$$\mathfrak{B}^G_S = \langle \mathrm{C}(G/H) \otimes \mathbb{M}_{S^\infty} \mid \text{cyclic } H \subseteq G \rangle$$

consists of precisely of the S-divisible objects in the equivariant bootstrap class $\mathfrak{B}^G$ in $\mathfrak{KK}^G$.

The UCT

Assumption

From now on, let S be the (finite) set of primes that divide the order $|G|$ of G.

Theorem (Meyer–N. 2024)

The functor

$$F\colon \mathfrak{KK}^G_S \to \mathfrak{A}[|G|^{-1}]$$

is the universal Abelian approximation. If $A, B \in \mathfrak{KK}^G_S$ and A belongs to the equivariant bootstrap class in $\mathfrak{KK}^G$, then there is a Universal Coefficient Theorem

$$\mathrm{Ext}_{\mathfrak{A}[|G|^{-1}]}(F(A), F(\Sigma B)) \rightarrowtail \mathfrak{KK}^G_S(A, B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{A}[|G|^{-1}]}(F(A), F(B)).$$

The functor F induces a bijection between isomorphism classes of S-divisible objects in the G-equivariant bootstrap class and isomorphism classes of objects in $\mathfrak{A}[|G|^{-1}]$.

No time left

Application: Kirchberg algebras with a finite group action

By a Kirchberg algebra we mean a nonzero, simple, purely infinite, nuclear C*-algebra.

Theorem (Gabe and Szabó 2024)

Any G-action on a separable, nuclear C-algebra is $\mathfrak{KK}^G$-equivalent to a pointwise outer action on a stable Kirchberg algebra. Two pointwise outer G-actions on stable Kirchberg algebras are $\mathfrak{KK}^G$-equivalent if and only if they are cocycle conjugate.*

Corollary (Meyer–N. 2024)

There is a bijection between the set of isomorphism classes of objects of $\mathfrak{A}[|G|^{-1}]$ and the set of cocycle conjugacy classes of pointwise outer G-actions on stable Kirchberg algebras that belong to the G-equivariant bootstrap class and are S-divisible in $\mathfrak{KK}^G$.

Example: inverting $|V|$ in $V = \mathbb{Z}/2 \times \mathbb{Z}/2$

Observe

Inverting $|V| = 4$ is the same as inverting 2.

We find that,

$$\mathfrak{Cyc}^{\mathrm{op}}[|V|^{-1}] \cong \mathbb{Z}[1/2]^{\times 10}$$

with 4 copies of $\mathbb{Z}[1/2]$ coming from the trivial subgroup and 3-double copies from three non-trivial cyclic subgroups. So,

$$\mathfrak{A}[|V|^{-1}] \cong \mathfrak{Mod}(\mathbb{Z}[1/2]^{\times 10})_{\mathrm{c}}.$$

Corollary

The isomorphism classes of 2-divisible objects in the V-equivariant bootstrap class in $\mathfrak{KK}^V$ are in bijection with 10-tuples of 2-divisible $\mathbb{Z}/2$-graded Abelian groups.

Thank you!