Lecture 9 - Abstract Vector Spaces

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Q: What are vectors?

Is a two dimentional vector $\vec{v} \in \mathbb{R}^2$ fundamentaly an arrow in 2D that we can discribe as a pair of numbers $\binom{x}{y}$ **or** is it fundamentally that pair of real numbers which is nicely visualized as an arrow in two dimentiontional flat plain?

We alredy know that any vector $\vec{v} \in \mathbb{R}^2$ can be represented by different pairs $\binom{x}{y}$ depending on a basis. So, it seems like we are dealing with objects that exist on their own, independent of their description by numbers in some basis. Core ideas, like determinants, eigenvectors are independent of our choice of basis.

But if vectors are **not** fundamentaly lists of real numbers, what are they?

Example: Consider the set of all functions from real numbers to real numbes denoted by $Map(\mathbb{R}, \mathbb{R})$.

- Note that we can add functions to get another function

$$(f+g)(x) := f(x) + g(x)$$

Very similar to adding vectors coordinate by coordinate, Very similar to adding vectors coordinate by coordinate, its just that there are infinitly many cooordinates.

- We can scale functions by number $r \in \mathbb{R}$.

$$(rf)(x) := r \cdot f(x)$$

Again, similar to multiplying vectors coordinatwize by scalars.

Since really only thing vectors in $\mathbb{R}^n$ could do is get added together and multiplied by a scalar, it seems like we should be able to take ideas from linear algebra (Linear transformation, kernel, dot product, eigenvalues, eigenvectors etc.) and apply it to functions $Map(\mathbb{R}, \mathbb{R})$.

For example derivative transforms one function into another $\frac{d}{dx} : Map(\mathbb{R}, \mathbb{R}) \longrightarrow Map(\mathbb{R}, \mathbb{R})$ (Sometimes, in this context, these transformations are called „operators").

Recall, that a transformation T is linear, if $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ and $T(k\vec{v}) = k\,T(\vec{v})$ for $k \in \mathbb{R}$.

But for a derivative, you have already been using these properties all the time, even if you have not specifically named them this way. That is,

- $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx}(k \cdot f) = k\frac{df}{dx}$.

- $\frac{d}{dx}(k \cdot f) = k \frac{df}{dx}$.

Subexample: $\frac{d}{dx}(2x^2 + 3x) = \frac{d(2x^2)}{dx} + \frac{d(3x)}{dx} =$

$= 2\frac{dx^2}{dx} + 3\frac{dx}{dx} = 4x + 3$

So, if functions are like vectors, derivative is an example of linear transformation!

But, we also know that we can describe linear transformations by a matrix; so, we should be able to describe $\frac{d}{dx}$ as a matrix multiplication.

Since $Map(\mathbb{R}, \mathbb{R})$ is „very" infinite dimentional, lets limit ourselves to $Pol(\mathbb{R}, \mathbb{R}) \subseteq Map(\mathbb{R}, \mathbb{R})$, where

$$Pol(\mathbb{R}, \mathbb{R}) = \left\{ \sum_{i=0}^{n} a_i x^i \;/\; a_i \in \mathbb{R} \right\}.$$

polynomials in one variable.

Subexample: $x^2 + 3x + \pi$; $2x^{100} + \frac{1}{2}x^{71} + \sqrt{2}x^3$.

The first thing do do, is do give coordinates to elements of $Pol(\mathbb{R}, \mathbb{R})$. Since polynomials are already written as a sum of other, „pure powers of x" polynomials, we can choose these pure power polynomials as a basis:

$$x^{15} + 2x^2 + 3x = 1 \cdot x^{15} + 2 \cdot x^2 + 3 \cdot x$$

already written as a linear combination of (already written as a linear combination of $\{1, x, x^2, x^3, x^4, \dots\}$

So, our basis functions are:

$$b_0(x) = 1 \quad , \quad b_1(x) = x, \quad b_2(x) = x^2, \dots, \quad b_n(x) = x^n, \dots$$

First, note that the basis we chose

$$\mathcal{B} = \{ b_0, b_1, b_2, b_3, \dots, b_n, \dots \}$$

has infinitely many elements, which means our functions will have infinite coordinates and oure "space" will be infinite dimentional. Then $x^{15} + 2x^2 + 3x$ has coordinates

$$x^{15} + 2x^2 + 3x \rightsquigarrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\sqrt{2}x^5 - 5x^3 \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ -5 \\ 0 \\ \sqrt{2} \\ 0 \\ \vdots \end{pmatrix}$$

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \rightsquigarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

To write down a matrix A of $\frac{d}{dx}$, recall that we need to write down coordinates of $\frac{d\, b_j(x)}{dx}$ and put them as columns of A. Computing that we need to write down coordinates of $\frac{\quad}{dx}$ and put them as columns of A. Computing,

$$\frac{d\,b_0(x)}{dx} = \frac{d\,1}{dx} = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \frac{d\,b_1(x)}{dx} = \frac{dx}{dx} = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\frac{d\,b_2(x)}{dx} = \frac{d\,x^2}{dx} = 2x = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \end{pmatrix}, \quad \frac{d\,b_3(x)}{dx} = \frac{d\,x^3}{dx} = 3x^2 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ \vdots \end{pmatrix}, \dots$$

So,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and, for example, finding $\frac{d}{dx}\left(x^2 + 4x^3 + 2x\right)$ amounts to

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 2 & \cdots & \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 12 \\ 0 \\ \vdots \end{pmatrix} \rightsquigarrow 2 + 2x + 12x^2$$

So, we found that seemingly different concepts of matrix multiplication and derivative are much the same from Linear Algebra perspective.

So, what is a vector? There are many „vector like"

So, what is a vector? There are many „vector like" things in mathematics, like arrows in 2D, pairs of numbers or functions from $\mathbb{R}$ to $\mathbb{R}$. We would like to apply all the concepts and theorems proved so far to all such „vector like" things. We call these things „vector spaces" in general, so that we do not have to think about all possible examples in separate. What we do, is establish list of rules that our vector addition and scalar multiplication has to abide by, and define a general vector space.

Definition: A vector space over a field $\mathbb{F}$ (this can be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_{p^n}$, $\mathbb{Q}$ etc.) is a set V, together with operations of addition of elements „$+$" and multiplication by the elements of $\mathbb{F}$ „$\cdot$", such that for all $u, v, w \in V$ and $a, b \in \mathbb{F}$

$$\left.\begin{array}{l} 1)\ u+(v+w)=(u+v)+w \\ 2)\ u+v=v+u \\ 3)\ \text{There is a vector } 0 \text{, such that} \\ \quad v+0=v \text{ for all } v\in V \\ 4)\ \text{For every } v \text{, there is a} \\ \quad \text{vector } -v \text{, so that } v+(-v)=0 \end{array}\right\} (V,+) \text{ is an abelian group!}$$

5) $a(b \cdot v) = (ab) \cdot v$

5) $a(b\cdot v) = (ab)\cdot v$

6) $1\cdot v = v$

7) $a(v+w) = a\cdot v + a\cdot w$

8) $(a+b)v = a\cdot v + b\cdot v$

So, now any set that can be endowed with a vector space structure over some field $\mathbb{F}$ (satisfies 8 axioms above) can be studied with results from linear algebra! Therefore, once we prove our results in terms of these axioms, we never have to discuss all concrete examples of vector spaces, it automatically applies to all. So, a vector is just an element of a vector space.

Example: 1) $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, with coordinatwise addition and scalar multiplication. $(0, 0, \dots, 0)$ is the neutral element (zero vector)

2) $Map(\mathbb{R}, \mathbb{R})$ is a vector space over $\mathbb{R}$ with scalar multiplication and addition defined above. Zero vector Θ is a function $\Theta(x) = 0$ for all $x \in \mathbb{R}$.

3) $Mat_{n\times m}(\mathbb{R})$ - the set of $n\times m$ matrices with real number entries is a vector space over $\mathbb{R}$. Zero matrix is a zero vector.

is a zero vector.

4) $\mathbb{R}^{\mathbb{N}}$ - the set of infinite sequences of real numbers is a vector space over $\mathbb{R}$, where addition and scalar multiplication are defined term by term:

$$(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) = (x_0+y_0, x_1+y_1, \dots)$$

$$k(x_0, x_1, x_2, x_3, \dots) = (kx_0, kx_1, kx_2, \dots)$$

zero vector is the sequence

$$(0, 0, 0, 0, \dots)$$

5) All the examples above will be vector spaces over $\mathbb{F}$ $(= \mathbb{Q}, \mathbb{Z}_2, \mathbb{C}, \mathbb{Z}_{p^n}, \dots)$ if we replace $\mathbb{R}$ by $\mathbb{F}$ everywhere.

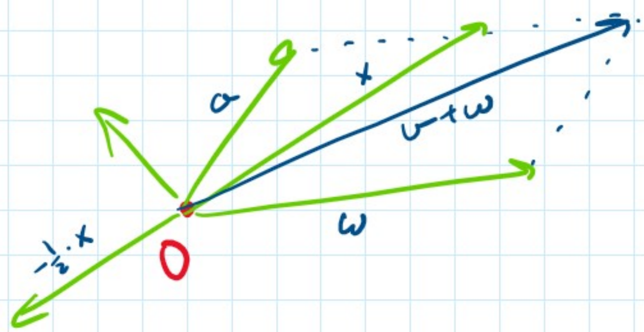
6) The linear equations in 3 unknowns

$$ax + by + cz = d$$

where a, b, c, d are constants in some $\mathbb{F}$ (again $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \dots$) form a vector space under addition and scalar multiplication (familiar operations of Gaussian elimination).

7) P - set of geometric vectors on the plane originating from a single point O. Addition by parallelogram law and scalar multiplication by scaling. P is a vector space over $\mathbb{R}$.

scaling. P is a vector space over $\mathbb{R}$.



By introduction of coordinate system, we can identify P with $\mathbb{R}^2$. This is the idea behind Descarte's Analytic Geometry.

8) Complex numbers $\mathbb{C}$ is a vector space over $\mathbb{R}$.

9) Complex numbers $\mathbb{C}$ is a vector space over $\mathbb{C}$. (By example 1) when we replace $\mathbb{R}$ with $\mathbb{C}$).

10) Consider the differential equation

$$f''(x) = -f(x)$$

Now, clearly $\sin(x)$ and $\cos(x)$ are solutions of this equation. Note that if $f_1(x)$ and $f_2(x)$ are solutions and $f(x) = f_1(x) + f_2(x)$ we have

$$f''(x) = f_1''(x) + f_2''(x) = -f_1(x) - f_2(x) = -f(x)$$

So, $f(x)$ is also a solution.

Likewise, if $f_1(x)$ is a solution and k is any scalar, Likewise, if $f_1(x)$ is a solution and k is any scalar, then $f(x) = k \cdot f_1(x)$ is also a solution. It follows that all linear combinations → (officially defined below)

$$f(x) = c_1 \sin(x) + c_2 \cos(x)$$

are solutions and the set of all solutions forms a vector space over $\mathbb{R}$ and actually has a basis $\{\sin(x), \cos(x)\}$!

Def: We say that element v of vector space over $\mathbb{F}$ is a linear combination of elements $v_1, \dots, v_n$ if

$$v = c_1 v_1 + \dots + c_n v_n \quad c_i \in \mathbb{F}.$$

Remark: „Linear space" and „Vector space" are synonyms.

Subspaces

A subset W of a linear space V is called a *subspace* of V if

- **a.** W contains the neutral element 0 of V.
- **b.** W is closed under addition (if f and g are in W, then so is $f + g$).
- **c.** W is closed under scalar multiplication (if f is in W and k is a scalar, then kf is in W).

We can summarize parts b and c by saying that W is closed under linear combinations.

Span, linear independence, basis, coordinates

Consider the elements $f_1, \dots, f_n$ in a linear space V.

a. We say that $f_1, \dots, f_n$ *span* V if every f in V can be expressed as a linear combination of $f_1, \dots, f_n$.

b. We say that f_i is *redundant* if it is a linear combination of $f_1, \dots, f_{i-1}$. The elements $f_1, \dots, f_n$ are called *linearly independent* if none of them is redundant. This is the case if the equation

$$c_1 f_1 + \cdots + c_n f_n = 0$$

has only the trivial solution

$$c_1 = \cdots = c_n = 0.$$

Span, linear independence, basis, coordinates ***(Continued)***

c. We say that elements $f_1, \dots, f_n$ are a *basis* of V if they span V and are linearly independent. This means that every f in V can be written uniquely as a linear combination $f = c_1 f_1 + \cdots + c_n f_n$. The coefficients $c_1, \dots, c_n$ are called the *coordinates* of f with respect to the basis $\mathfrak{B} = (f_1, \dots, f_n)$. The vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

in $\mathbb{F}^n$ is called the $\mathfrak{B}$-*coordinate vector* of f, denoted by $[f]_{\mathfrak{B}}$. The transformation

$$L(f) = [f]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{from } V \text{ to } \mathbb{F}^n$$

is called the $\mathfrak{B}$-*coordinate transformation,* sometimes denoted by $L_{\mathfrak{B}}$.

$\mathfrak{B}$-coordinate transformation is invertible, with inverse

$\Big(c_1$

inverse

$$L^{-1}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

In particular, $L^{-1}(e_i) = f_i$.

Dimension

If a linear space V has a basis with n elements, then all other bases of V consist of n elements as well. We say that n is the *dimension* of V:

$$\dim(V) = n.$$

Proof: Analogous to $\mathbb{R}^n$, exercise.

A Vector space over $\mathbb{F}$ is called finite dimentional if it has a (finite) basis $f_1, \dots, f_n$, so that we can define its dimention $\dim(V) = n$. Otherwise it is called infinite dimentional.

Example: 1) $Pol(\mathbb{R}, \mathbb{R})$ is infinite dimentional.

2) linear equations in 3 unknows is finite dimentional (Why?)

3) Let V be the solution vector space of differential equation $f''(x) = -f(x)$. We will prove that V is 2-dimentional as $\{\sin(x), \cos(x)\}$ forms a basis of V. We already know that

a basis of V. We already know that $\sin(x), \cos(x) \in V$. Clearly $\sin(x) \neq k \cdot \cos(x)$ for any $k \in \mathbb{R}$, so we only need to show that $\operatorname{span}(\sin(x), \cos(x)) = V$.

Now, $\operatorname{span}(\sin(x), \cos(x)) \subseteq V$ obviously. Say now $g(x) \in V$. Then

$$\left(g(x)^2 + (g'(x))^2 \right)' = 2\, g(x) \cdot g'(x) + $$
$$+ 2 g'(x) \cdot \underbrace{g''(x)}_{= g(x)} =$$
$$= 2\, g(x) \cdot g'(x) - 2 g'(x) \cdot g(x) = 0$$

So, $g(x)^2 + g'(x)^2$ is constant.

Now, assume $g(0) = g'(0) = 0$, then

$$g(x)^2 = -g'(x)^2 + c \implies g(0)^2 = -g'(0)^2 + c \implies$$
$$\implies c = 0 \implies g(x) = -g'(x) \implies$$
$$\implies g'(x) = -g''(x) = g(x) = -g'(x)$$
$$\implies g'(x) = 0 \text{ for all } x. \implies$$
$$\implies g(x) \text{ constant } \quad g(0) = 0 \implies g(x) = 0$$

Now for any $f(x) \in V$,

$$g(x) = f(x) - f(0) \cos x - f'(0) \sin x \in V.$$

$g(0) = f(0) - f(0) \ldots 0$

$$g(0) = f(0) - f(0) = 0$$

$$g'(0) = f'(0) - f'(0) = 0$$

Thus $g(x) = 0$, so

$f(x) = f(0)\cos x + f'(0)\sin x$, therefore

$\{\cos x, \sin x\}$ is a basis of V.