Lecture 8 - Complex numbers II

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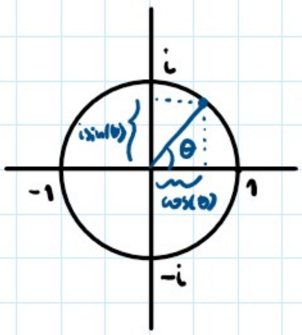
In the last lecture, we found out that complex numbers $z \in \mathbb{C}$ with $|z| = 1$ play a special role as they represent pure rotations. Such complex numbers have a polar form $z = \cos\theta + i\sin\theta$.



These complex numbers can also be represented as $e^{i\theta}$, where e is an Euler's number. That is, we have a famous Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Q: What does $e^{i\theta}$ denote?

First, we go through a small recollection session from Calculus. Recall the corollary of Taylor's theorem:

Corollary (of Taylor's thm): Let a function $f: \mathbb{R} \to \mathbb{R}$ be infinitely differentiable at point 0. Then

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{(Proof in CE8)}$$

for some constants $a_k \in \mathbb{R}$. $\square$

Say now $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then we have

$$f(0) = a_0, \quad f'(0) = \left(\sum_{k=0}^{\infty} a_k x^k\right)'_{x=0} = \left(\sum_{k=0}^{\infty} k \cdot a_k \cdot x^{k-1}\right)_{x=0} = a_1$$

$$f''(0) = \left(\sum_{k=0}^{\infty} k a_k x^{k-1}\right)'_{x=0} = \left(\sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}\right)_{x=0} = 2a_2$$

$$f'''(0) = \left(\sum k(k-1) a_k x^{k-2}\right)' = \left(\sum k(k-1)(k-2) a_k x^{k-3}\right) =$$

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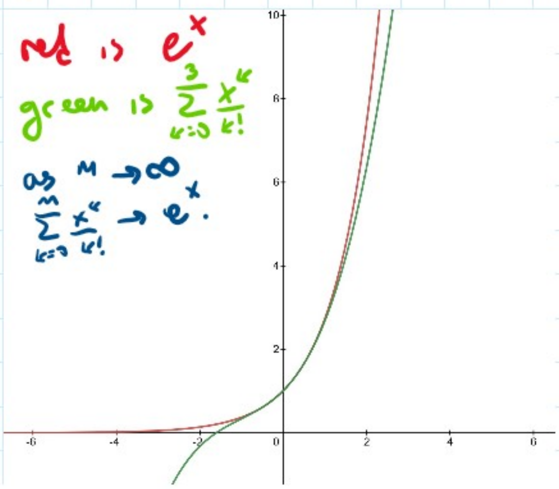
$$= 3\cdot 2\, a_3$$

and in general $f^{(k)}(0) = k!\, a_k$. So, we conclude that
$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}$$

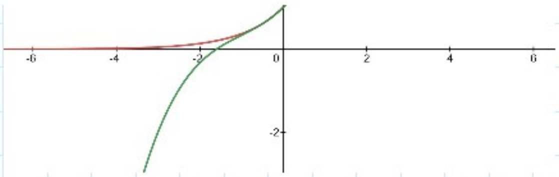
Example: 1) For $f(x) = e^x$, we have $(e^x)' = e^x$ and is thus infinitely differentiable everywhere. Now,
$$f(0) = e^0 = 1, \quad f'(0) = f''(0) = \dots = f^{(k)}(0) = e^0 = 1.$$
Therefore $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

The function $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is denoted by $\exp(x)$

Remark: In general, $\exp(x)$ and e^x are different things, $e^2 := e\cdot e$ and $\exp(2) := 1 + 2 + \frac{4}{2!} + \frac{8}{3!} + \dots$ but $e^x = \exp(x)$ as functions on $\mathbb{R}$. In Eulers formula, $e^{i\theta}$ actually means $\exp(i\theta)$!



In particular, since $\exp(x) = e^x$, we have that $\exp(a+b) = \exp(a)\exp(b)$ Which, even though not obvious, can be directly shown using definition of exp! (in CE) This property alone is very restrictive! It already defines exp as a power function.



already defines exp as a power function:

$$\exp(n) = \exp(\underbrace{1+\dots+1}_{n}) = \exp(1)^n \text{ for } n \in \mathbb{N}.$$

$$\exp\left(\tfrac{1}{n}\right)^n = \underbrace{\exp\left(\tfrac{1}{n}\right)\cdot \exp\left(\tfrac{1}{n}\right)\cdots}_{n} \exp\left(\tfrac{1}{n}\right) = \exp\Big(\underbrace{\tfrac{1}{n}+\dots+\tfrac{1}{n}}_{n}\Big) = \exp(1) \implies$$

$$\implies \exp\left(\tfrac{1}{n}\right) = \sqrt[n]{\exp(1)}$$

if $\exp \neq 0$, $\exp(x) = \exp(x+0) = \exp(x)\cdot\exp(0) \Rightarrow \exp(0) = 1.$

$$\exp(-n)\cdot\exp(n) = \exp(-n+n) = \exp(0) = 1 \implies \exp(-n) = \frac{1}{\exp(1)}$$

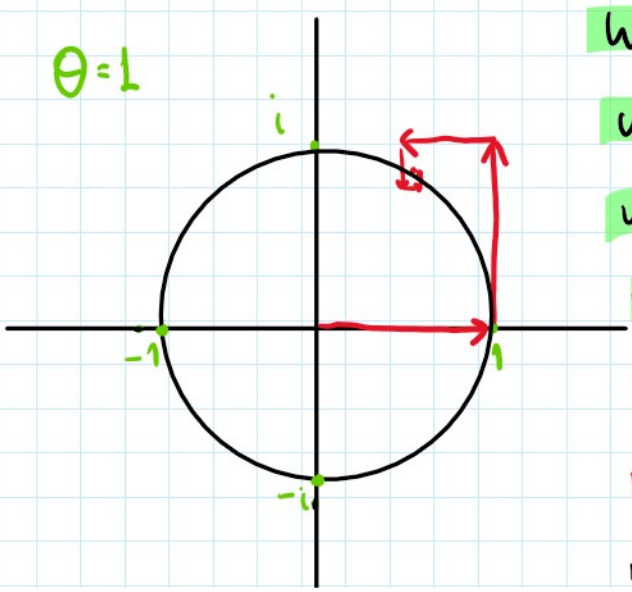
$$\exp\left(\tfrac{p}{q}\right)^q = \exp\left(q\cdot\tfrac{p}{q}\right) = \exp(p\cdot q) = \exp(1)^{pq} \Rightarrow \exp\left(\tfrac{p}{q}\right) = \exp(1)^{\frac{p}{q}}$$

In general, letter e is just a notation for a number $\exp(1)$.

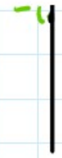
In other word, we can also just define $e := \sum_{k=0}^{\infty} \frac{1}{k!}$

Remark: Note, that given certain set of objects such that we can divide them by numbers, add them, multiply them (think matrices, complex numbers etc.) then we can define exp function on those objects.

In particular, this means that since we can compute any polynomials of complex numbers, we can plug in complex number $i\theta \in \mathbb{C}$, $\theta \in \mathbb{R}$: $\exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$



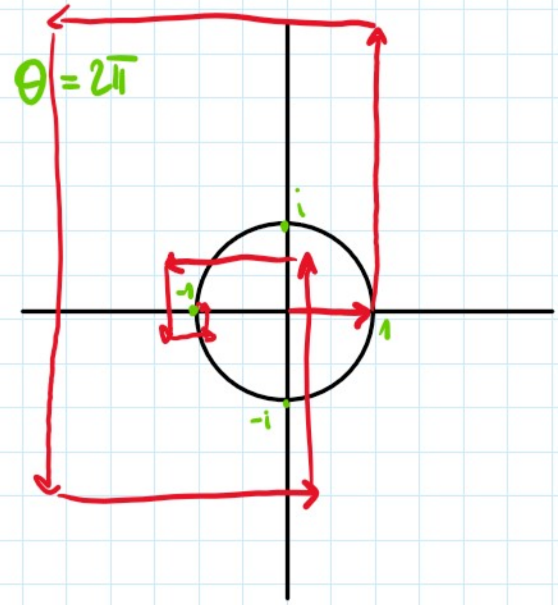
What Eulers formula tells us, is that when we do this, we will always end up on a unit circle! This fact is not trivial, and is not just a matter of notation. For example if $\theta = \pi$, by Euler formula



if $\theta = \pi$, by Euler formula we will get $e^{i\pi}$ $(\exp(i\pi)) =$

$$= 1 + \pi i + \frac{(\pi i)^2}{2!} + \frac{(\pi i)^3}{3!} + \dots =$$

$$= \cos\pi + i\sin\pi = -1.$$



In addition, Eulers formula tels us that $e^{i\theta}$ is a periodic function (since cos and sin are periodic) with period 2π when we plug in purely imaginary $(i\theta)$ numbers. So, the situation is the following, we have a function $\exp(x)$, which when considered on real numbers is a power function $\exp(x) = \exp(1)^x$ (we call $\exp(1) =: e$) but when we plug in imaginary numbers (of the form $i\theta$) we get a periodic function with period 2π.

Proof of Euler formula: Lets compute Taylor series for $\sin(x)$ and $\cos(x)$. We get:

1) For $f(x) = \cos x$, $f'(x) = -\sin(x)$ and thus it is again differentiable everywhere. We have, $f(0) = \cos(0) = 1$.
$f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, $f^{(4)}(0) = 1$ and in general: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

in general: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

2) For $f(x) = \sin(x)$, $f'(x) = \cos(x)$ and we get

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

We can also plug in complex numbers into these polynomials

Also, adding $\cos x + i \sin x = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$

$$= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = \exp(ix). \ \square.$$

Remark: By Eulers formula we have, for $x \in \mathbb{R}$

$$e^{-ix} = e^{i(-x)} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$$

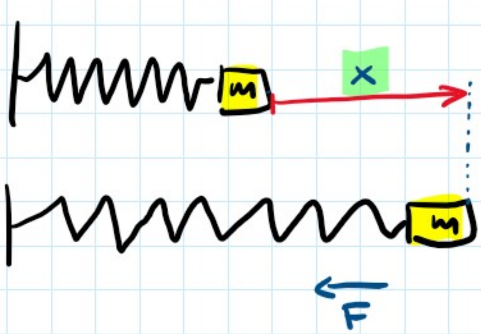
So, $e^{-ix} + e^{ix} = \cos x + i \sin x + \cos x - i \sin x = 2 \cos x$

therefore $\cos x = \frac{e^{-ix} + e^{ix}}{2}$. Similarly, $e^{ix} - e^{-ix} =$

$= \cos x + i \sin x - \cos x + i \sin x = 2i \sin x$, and therefore

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. This identities make proving trigonometric identities much easier!

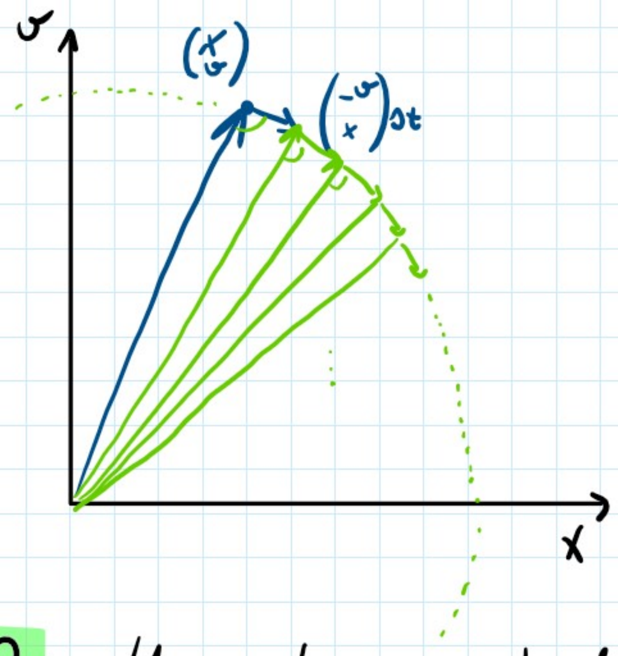
Insight and Example: Hooke's law



Say we are given a spring with mass m attached to its end. Hooke's law tells us that if we stretch the spring

$\overleftarrow{F}$ that if we stretch the spring by amount x, then the pulling force F from the spring is linearly proportional to F. That is $F = -k \cdot x$, where k is a proportionality costant (depends on a spring). By Newtons second law $F = ma$, so, we have $ma = -kx$, or $a = -\frac{k}{m}x$.

Lets assume that whatever units we are working with, $\frac{k}{m} = 1$, so, we just have $a = -x$.



We will represent our situation on a 2D plane, where each point represent a pair $\binom{x}{v}$ of the „state" of our spring, wher x is displacement and v is ten velocity of the spring at that displacement.

Q: How do coordinates $\binom{x}{v}$ change over small period of time Δt?

change of displacement is $v \cdot \Delta t$

This is a $90°$ rotation of $\binom{x}{v}$! in clockwise dir.

$$\Delta \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \cdot \Delta t \\ a \cdot \Delta t \end{pmatrix} = \begin{pmatrix} v \cdot \Delta t \\ -x \cdot \Delta t \end{pmatrix} = \begin{pmatrix} v \\ -x \end{pmatrix} \cdot \Delta t$$

change of velocity is $a \cdot \Delta t$

But now, instead of writting $\binom{x}{v}$, lets write

But now, instead of writing $\binom{x}{v}$, lets write it as a complex number $x+iv$. Then,

$\Delta(x+iv) = -i\cdot(x+iv)\Delta t = (v - xi)\Delta t$. Or, in general, for $z = x+iv$, Hooke's law can be described as $\Delta z = -i\, z\cdot\Delta t$, and we go around the circle as we compound infinitly small Δt's (see picture). This also makes intuitive sense, as velocity is increasing, displacement is decreasing and vice-versa.

Lets compute how our complex numbers are compounding: Say we start at time $t=0$, with original complex number z_0, After Δt time we have

$$z_1 = z_0 + \Delta z_0 = z_0 - i z_0 \cdot \Delta t = z_0(1 - i\Delta t), \text{ then}$$

$$z_2 = z_1 + \Delta z_1 = z_1 - i z_1 \Delta t = z_0(1-i\Delta t) - i z_0(1-\Delta t)\Delta t =$$
$$= z_0(1-i\Delta t)(1-i\Delta t) = z_0(1-i\Delta t)^2$$

In general, after time T, if we took n Δt 'steps' we get

$$z_T = z_0(1-i\Delta t)^n = z_0\left(1 - i\frac{T}{n}\right)^n = \text{(by Binomial formula)}$$
$$= z_0\left(1 - iT + \frac{n(n-1)}{2!\,n^2}\cdot(iT)^2 - \frac{n(n-1)(n-2)}{3!\,n^3}(iT)^3 + \dots\right)$$

which, when $n \to \infty$ (steps become smaller) is equal to

$$= z_0\left(1 + (-iT) + \frac{(-iT)^2}{2!} + \frac{(-iT)^3}{3!} + \dots\right) =$$
$$= z_0 e^{-iT}.$$

So, $z_T = z_0 e^{-iT}$! ← rotation on a circle

$= z_0 e$. So, $Z_T = z_0 e$! ← a circle

By Eulers formula: $Z_T = z_0 (\cos(T) - i\sin(T))$

What it is saying is very intuitive: the displacement part of our complex number is $z_0 \cdot \cos(T)$ and velocity part is $-z_0 \sin(T)$. They oscilate and are out of sink!