Lecture 7 - Complex numbers I

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We start by the observation that we have always gradually expended sets of numbers considered by adding „solutions" to different equations:

$x + 1 = 0 \implies$ „-1" a solution

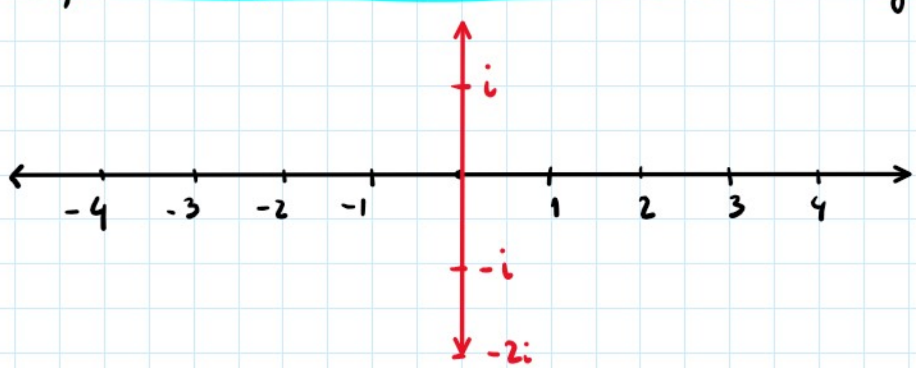
$1 \cdot x = 2 \implies$ „$\frac{1}{2}$" a solution

$x^2 = 2 \implies$ „$\sqrt{2}$" a solution

Similarly, we will now add a „solution" to the equation

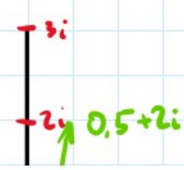
$$x^2 = -1$$

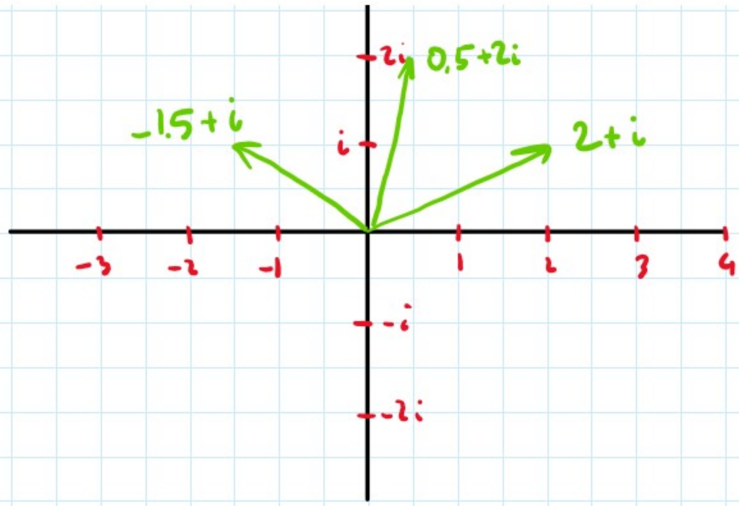
So, we make an assumption: There is a number i, so that $i^2 = -1$. Let us also make an assumption that on a coordinate axis, numper i is perpendicular to this axis, intersects at zero and has length 1.



Q: Why make these assumptions?

Befor justifying these assumptions, lets see how can we define addition for such 2D numbers:

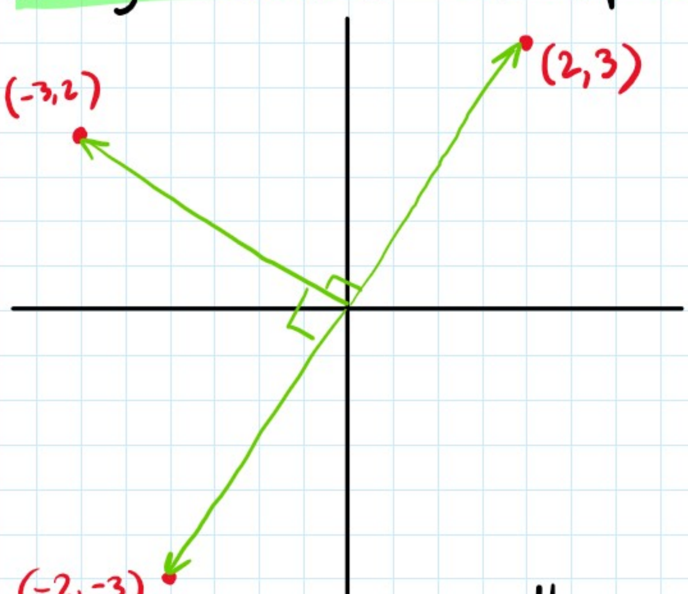




We use vector addition, as if our vectors are written in $(1, i)$ basis. So, then

$$(2+i)+(-1.5+i) = 0.5+2i$$

The interesting part here is that we can also define very nicely behaved multiplication!



Take some point with coordinates, say, $(2,3)$, and lets see what happens when we rotate it coun. clockwise by $90°$ (recall that this is just mult. by $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$).

$$(2,3) \xrightarrow{90^\circ \text{rot.}} (-3,2)$$

generally $(x,y) \xrightarrow[90^\circ \text{rot.}]{\begin{pmatrix}0&-1\\1&0\end{pmatrix}} (-y,x)$

and continuing,

$$\begin{tikzcd} (x,y) \arrow[r, "90^\circ \text{rot}"] \arrow[rr, bend right, "180^\circ \text{rot.}"'] & (-y,x) \arrow[r, "90^\circ \text{rot}"] & (-x,-y) \end{tikzcd}$$

Now, lets „mechanistically" calculate $(2+3i)\cdot i$ product, Now, lets „mechanistically" calculate $(2+3i)\cdot i$ product, assuming distributivety, commutativety and $i^2=-1$ holds:

$$(2+3i)\cdot i = 2\cdot i + 3\cdot i^2 = -3+2i$$

So, geometrically multiplication by vector i rotates by $90°$ coun. clockwise, so maybe positioning i perpendic. to x-axis was not a bed idea ($1\cdot i = i$ rotation by $90°$)

This idea would allow us to „define" multiplication by any 2D number z. But, Q: How? We assume 3 things:

1) $z\cdot 1 = z$. 2) $z\cdot i$ = Rotation $90°$ of z.

3) $z\cdot(c+di) = c\cdot z + d(iz)$ – distributivety.

So, by distributivety, when multiplieng z by $c+di$, we scale z by c, and add $90°$ rotated z scaled by d. So, knowing these 3 rules, completely determines multiplication of two 2D numbers.

Example: $(1+i)\cdot(2-2i)$. Geometrically, we need to add $2(1+i)$ to $-2\cdot(-1+i)$ Rot$90°(1+i)$, or $2+2i+2-2i = 4$. Also, algebraically,

$$(1+i)(2-2i) = 1\cdot 2 - 1\cdot 2i + i\cdot 2 - 2(i)^2 = 4$$

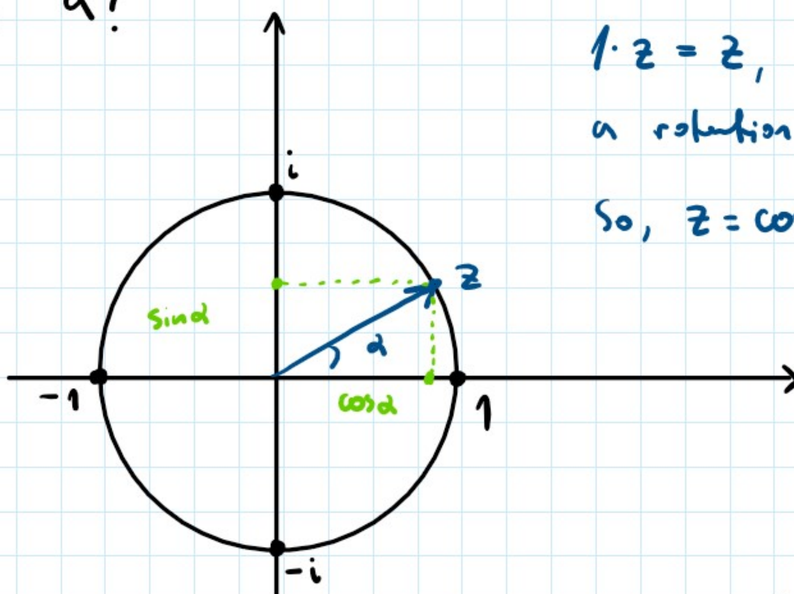
Geometry: Multiplication is rotation and stretching.

More precisely, any real numer $r\in\mathbb{R}$, as an element of the group $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ can be seen as „stretching" by a factor of „r" (stretch number line so that neutral elem 1 goes to r) the group $(\mathbb{R}\setminus\{0\}, \cdot)$ can be seen as "stretching" by a factor of "r" (stretch number line so that neutral elem. 1 goes to r)

Similarly, multiplication by 2D number z is stretching + rotation: stretching of plane so that neut. element 1 go to z. (See links)

Definition: numbers $a+bi$ with $a, b \in \mathbb{R}$, and $i^2 = -1$, with multiplication and addition defined above are called complex numbers and are denoted by $\mathbb{C}$.

Q: What is the complex number z, multiplication by which is a rotation by α?



$1 \cdot z = z$, so z is a rotation of 1 by α.

So, $z = \cos\alpha + i\sin\alpha$

Examples of uses: How do we calculate $\cos(75°)$?

$\cos(75°) = \cos(45° + 30°)$. We use complex numbers:

Rotation by $45° = \cos(45°) + i\sin(45°) =: z$

Rotation by $30° = \cos(30°) + i\sin(30°) =: w$

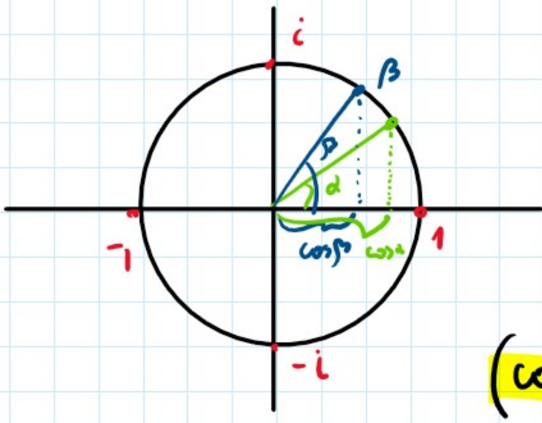
Rotation by $75° = z \cdot w$

$$z \cdot w = (\cos 45° + i\sin 45°)(\cos 30° + i\sin 30°) =$$

$$= \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) =$$

$$= \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2} \right) =$$

$$= \underbrace{\left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \right)}_{\cos 75^\circ} + i\left(\ldots \right).$$ So, in general



$$\overbrace{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}^{\text{rotation by } \alpha+\beta} =$$

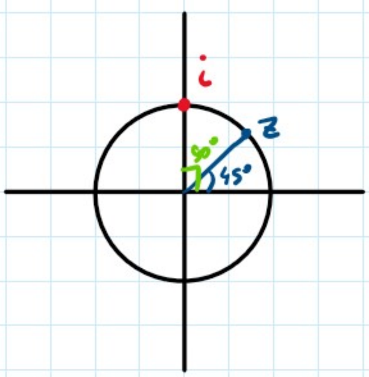
$$= \underbrace{(\cos\alpha + i\sin\alpha)}_{\text{rot by } \alpha} \underbrace{(\cos\beta + i\sin\beta)}_{\text{rot by } \beta}$$

Multiplying right hand side:

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\alpha) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

and we derive the familiar angle relations.

Q: Does there exist $z \in \mathbb{C}$, s.t. $z^2 = i$?



Geometrically, we see that if we are looking for z, which when we rotate again by the angle z makes with x axis, we get 90° (that is, i).

So, we see that, for example

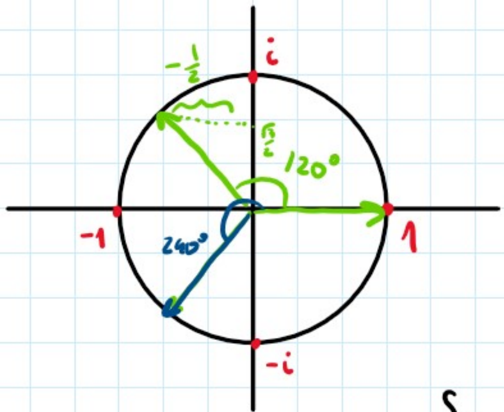
$z = \cos 45^\circ + i\sin 45^\circ = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$. Checking:

$$z^2 = \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2} \right)\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2} \right) = \left(\frac{2}{4} - \frac{2}{4} \right) + \left(\frac{2}{4} + \frac{2}{4} \right)i = i$$

Note, that we can also rotate by $(45^\circ + 180^\circ)$ so, for $w = \cos 225^\circ + i\sin 225^\circ$, $w^2 = i$. In general, so, for $w = \cos 225 + i \sin 225$, $w^2 = i$. In general, every complex number, exept zero, will have two distinct square roots. (what are two roots of -1?)

Q: Find 3 solutions to the equation $x^3 = 1$

(Note: there are only 3 solutions!).



Clearly $1^3 = 1$, that is rotating $z_1 = 1$ 3-times by $0°$ is again 1. Another angle which taken 3-times gives $360°$ (that is $0°$) is $120°$.

So, $z_2 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

So, $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = 1$. Another such angle is $240°$.

So, $z_3 = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ and $z_3^3 = 1$.

A polar form

The length of the vector $\binom{a}{b}$ is called the absolute value (norm, magnitude) of the complex number $z = a + ib$ and is denoted by $|z|$. By Pythagoras theorem $|z| = \sqrt{a^2 + b^2}$.

We saw that any $z \in \mathbb{C}$ which lies on a unit circle (that is, $|z| = 1$) is of the form $z = \cos\varphi + i \sin\varphi$ and represent a rotation by angle φ. Now, for any complex number $a + bi$ and $r \in \mathbb{R}^+$, $r(a + bi) =$

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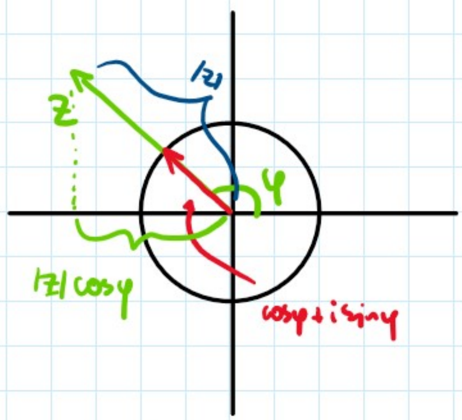
$= ra + rbi$ is stretching of vector $\binom{a}{b}$ by factor r.
So, any complex number z, that creates angle φ with x-axis can be written as stretching of vector $\cos\varphi + i\sin\varphi$ by factor $|z|$. So, any $z \in \mathbb{C}$, we have $z = |z|(\cos\varphi + i\sin\varphi)$ – this is sometimes called trigonometric, or a polar form of a complex number.



Q: Given any complex number z, how to find z^n? $n \in \mathbb{Z}$.
First, consider $n=2$. Then
$z^2 = \big(|z|(\cos\varphi + i\sin\varphi)\big)\big(|z|(\cos\varphi + i\sin\varphi)\big)$
$= |z|^2(\cos 2\varphi + i\sin 2\varphi)$ (by previous calculations, since we are rotating by φ twice.) Formalizing intuition:

Proposition (De Moivre formula): For any integer $n \in \mathbb{Z}$, we have

$$z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$$

Proof: In Central Exercise $\square$

Corollary: The complex number $z = |z|(\cos\varphi + i\sin\varphi)$ has n distinct n^{th} roots given by $(n \in \mathbb{N})$

$$w_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right) \quad k = 0, 1, \dots, n-1.$$

(That is $w_k^n = z$ for each $k = 0, 1, \dots, n-1$)

Proof: Say, $w^n = z$ and $w = |w|(\cos\alpha + i\sin\alpha)$.

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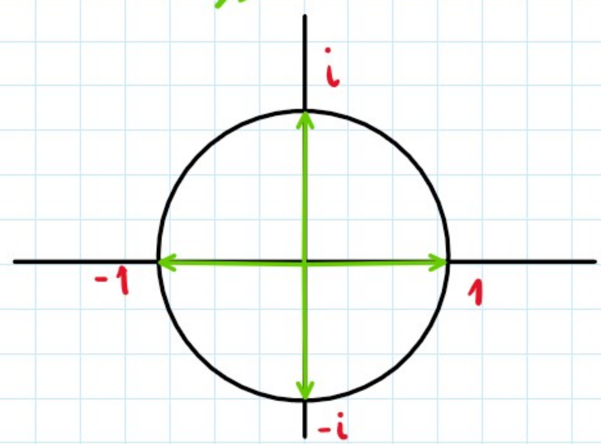
by De Moivre formula $w^n = |w|^n(\cos n\alpha + i\sin n\alpha)$. So,

$$|w|^n(\cos n\alpha + i\sin n\alpha) = |z|(\cos\varphi + i\sin\varphi), \text{ thus}$$

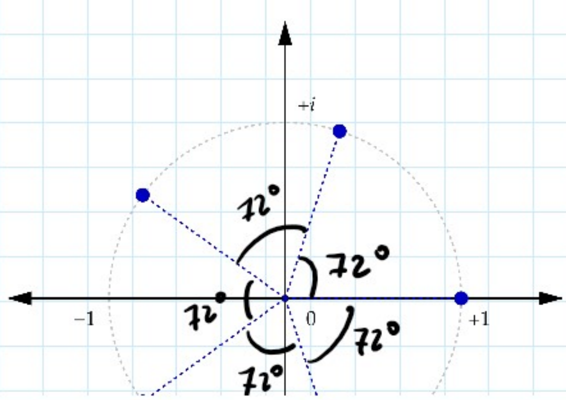
$|w|^n = |z|$, $\cos n\alpha = \cos\varphi$ and $\sin n\alpha = \sin\varphi$; so, $|w| = \sqrt[n]{|z|}$, $n\alpha = \varphi + 2\pi k$ for $k \in \mathbb{Z}$. Now, we only get n distinct solutions, since for $k = n$, $\alpha = \frac{\varphi + 2\pi k}{n} = \frac{\varphi}{n} + 2\pi$ and $\alpha = \frac{\varphi}{n}$ are „same" (co-terminal) angles. $\square$

Example: Find all 4^{th} roots of 1. We will have 4 roots, given by: $(1 = 1(\cos 0° + i\sin 0°))$

$$z_0 = \cos 0° + i\sin 0° = 1$$
$$z_1 = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i$$
$$z_3 = \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4} = -1$$
$$z_4 = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = -i$$

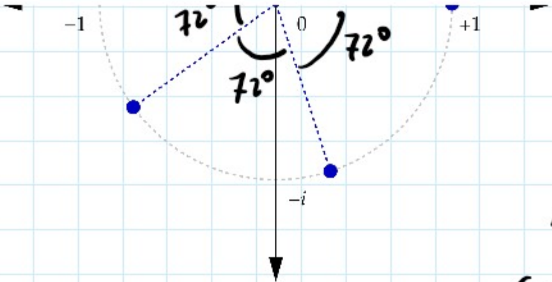


A complex number $u \in \mathbb{C}$ such that $u^k = 1$ for some $k \in \mathbb{N}_0$ is called a k^{th} root of unity.



For example, picture on the left shows all 5^{th} root of unity.

Now, the product of two



Now, the product of two n^{th} roots of unity, say, z and w is again n^{th} root of unity:

$(z \cdot w)^n = z^n \cdot w^n = 1 \cdot 1 = 1$. Also, if $z = \cos\varphi + i\sin\varphi$, then $u = \cos(2\pi - \varphi) + i\sin(2\pi - \varphi)$ is also the n^{th} root of unity and $z \cdot u = 1$. Clearly, 1 is also n^{th} root of unity. So, Set of n^{th} roots of unity, denoted by $\mathcal{R}(n)$ form a subgroup of $(\mathbb{C}, 1, \cdot)$ under multiplication.

In the end, we also give a very usful definition:

Definition: Let $z = a + ib$ be a complex number. We define $\bar{z} = a - ib$. The complex number $\bar{z}$ is called the conjugate of z.

Example: $\overline{3 + 2i} = 3 - 2i$. Geometrically, conjugation is a reflection by x-axis. Therfore, inverse of the root of unity w is $\bar{w}$.