Lecture 4 - Composition of linear transformations

Lets recap some relevant fact about $\mathbb{R}^n$.

- We can add any two vectors in $\mathbb{R}^n$. This additions turns $\mathbb{R}^n$ into an abelian group.

  * associative * neutral element * inverse * commutative

- We can multiply any vector in $\mathbb{R}^n$ by a scalar $a \in \mathbb{R}$.

  * $a(b \cdot \vec{v}) = (ab) \cdot \vec{v}$ * $1 \cdot v = v$ * $a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u}$

  * $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

Now, consider a function $T : \mathbb{R}^m \to \mathbb{R}^n$. Since linear algebra considers $\mathbb{R}^m, \mathbb{R}^n$ as sets of vectors, we ask T to preserve the structure of addition and scalar multiplication. That is, in linear algebra, we consider functions $T : \mathbb{R}^m \to \mathbb{R}^n$ such that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(a \cdot \vec{x}) = a \cdot T(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^m$ and $a \in \mathbb{R}$. These functions are, as we defined, linear transformations.

Recall our example about PageRank. Say we want to know the distribution of surfers after 2 iterations. This means we need to apply the transformation 2 times and compute the outcome.

Generally, say we have the situation with two transformations

$$\mathbb{R}^m \xrightarrow{T} \mathbb{R}^l \xrightarrow{S} \mathbb{R}^n$$

transformations

$$\mathbb{R}^m \xrightarrow{T} \mathbb{R}^p \xrightarrow{S} \mathbb{R}^n$$
$$\vec{x} \longmapsto T(\vec{x}) \longmapsto S \circ T(x) \; \left(:= S(T(x))\right)$$

Exercise: Show that $S \circ T$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Say a matrix of T is A and a matrix of S is B.

Definition: Product of matrices B and A, written as BA, is defined to be the matrix of the linear composition $S \circ T$.

As a matrix of a linear transformation $S \circ T : \mathbb{R}^m \to \mathbb{R}^n$ BA is $n \times m$ matrix.

Example: Say T is given by transformation

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3 \times 2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and S is given by

$$S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2 \times 3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$

$$S \circ T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S\begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} (x_1 + 2x_2) + 2 \cdot (3x_1 + 4x_2) + 3(5x_1 + 6x_2) \\ 4(x_1 + 2x_2) + 5 \cdot (3x_1 + 4x_2) + 6(5x_1 + 6x_2) \end{pmatrix}$$

$$\Big((1 + 2 \cdot 3 + 3 \cdot 5) x_1 + (2 + 2 \cdot 4 + 3 \cdot 6) x_2 \quad \Big)$$

$$= \begin{pmatrix} (1+2\cdot3+3\cdot5)x_1 + (2+2\cdot4+3\cdot6)x_2 \\ (4+5\cdot3+6\cdot5)x_1 + (4\cdot2+5\cdot4+6\cdot6)x_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+2\cdot3+3\cdot5 & 2+2\cdot4+3\cdot6 \\ 4+5\cdot3+6\cdot5 & 4\cdot2+5\cdot4+6\cdot6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 22 & 26 \\ 49 & 64 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So,

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2\times3} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3\times2} = \underbrace{\begin{pmatrix} 22 & 26 \\ 49 & 64 \end{pmatrix}}_{2\times2}$$

Generally, say B is $n\times p$ matrix and A is $p\times m$ matrix. Lets think about the columns of BA

$$\begin{aligned} (i\text{th column of } BA) &= (BA)\vec{e}_i \\ &= B(A\vec{e}_i) \\ &= B(i\text{th column of } A). \end{aligned}$$

If we denote the columns of A by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, we can write

$$BA = B \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}.$$

The columns of the matrix product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. Then, the product BA is

$$BA = B \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}.$$

$$BA = B\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Bv_1 & Bv_2 & \cdots & Bv_m \\ | & | & & | \end{bmatrix}.$$

To find BA, we can multiply B by the columns of A and combine the resulting vectors.

This is exactly what we did in the example above when computing the product of matrices.

Example: 1)

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) =$$

$$= \begin{pmatrix} 12 & 3 \\ 6 & 9 \end{pmatrix}$$

2)

$$\begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \left(\begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) =$$

$$= \begin{pmatrix} 11 & 4 \\ 5 & 10 \end{pmatrix}$$

From example we see that, in general, $AB \neq BA$. It might happen that $AB = BA$ for some particular A, B. In this case we say that A and B **commute** (Note, that necesserally A and B are both square matrices of the same dimention).

Now, the ij^{th} entry of the product BA is the i^{th} component of the vector $B\vec{v}_j$, which is the dot product of the i^{th} row of B with the j^{th} column of A.

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The entries of the matrix product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix. The ijth entry of BA is the dot product of the ith row of B with the jth column of A.

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ip} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \end{bmatrix}$$

is the $n \times m$ matrix whose ijth entry is

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ip}a_{pj} = \sum_{k=1}^{p} b_{ik}a_{kj}.$$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 2\cdot 5 & 1\cdot 3 + 2\cdot 0 \\ 3\cdot 2 + 0\cdot 5 & 3\cdot 3 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 6 & 9 \end{pmatrix}$

as before.

Remark: Matrix and vector multiplication is a special case.

Matrix Algebra

Matrix corresponding to the identity transformation $I: \mathbb{R}^n \to \mathbb{R}^n$ is clearly an identity matrix I_n.

For any $n \times m$ matrix A

$$AI_m = I_n A = A$$

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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Say A is $n \times p$ matrix, B is $p \times q$ and C is $q \times m$, then $(AB)C = A(BC)$ - that is, matrix product is associative.

Proof: Consider two linear transformations

$$T(\vec{x}) = \big((AB)C\big)\vec{x} \quad \text{and} \quad L(\vec{x}) = \big(A(BC)\big)\vec{x}$$

which are identical because, by definition of matrix multiplication

$$T(\vec{x}) = \big((AB)C\big)\vec{x} = (AB)(C\vec{x}) = A\big(B(C\vec{x})\big)$$

$$L(\vec{x}) = \big(A(BC)\big)\vec{x} = A\big((BC)\vec{x}\big) = A\big(B(C\vec{x})\big)$$

(That is, composition of functions is associative)

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "C"'] \arrow[rr, bend left, "BC"] \arrow[rrr, bend left=50, "A(BC)"] \arrow[rrr, bend right=50, "(AB)C"'] & \mathbb{R}^q \arrow[r, "B"'] \arrow[rr, bend right, "AB"'] & \mathbb{R}^p \arrow[r, "A"'] & \mathbb{R}^n
\end{tikzcd}$$

$\square$.

Distributive property for matrices

If A and B are $n \times p$ matrices, and C and D are $p \times m$ matrices, then

$$A(C + D) = AC + AD, \quad \text{and}$$
$$(A + B)C = AC + BC.$$

Proof: exercise.

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If A is an $n \times p$ matrix, B is a $p \times m$ matrix, and k is a scalar, then

$$(kA)B = A(kB) = k(AB).$$

Proof: exercise.

Inverse of the linear transformation

Recall that a function $f: X \to Y$ has an inverse, if and only if it is bijective, that is, for every $y \in Y$ there exists unique $x \in X$, such that $f(x) = y$. (every = surjective, unique = injective).

In latter case, there exist a function $f^{-1}: Y \to X$, such that $f \circ f^{-1} = f^{-1} \circ f = Id$.

For linear transformations this means

$T: \mathbb{R}^n \to \mathbb{R}^n$ has an inverse ($\mathbb{R}^n \to \mathbb{R}^m$ can never be a bijection unless $n = m$)

$\iff$ $T: \mathbb{R}^n \to \mathbb{R}^n$ is bijective

$\iff$ For every $\vec{y} \in \mathbb{R}^n$ there exists $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$.

$\iff$ $A\vec{x} = \vec{y}$ has a unique solution for every $\vec{y} \in \mathbb{R}^n$.
(A is $n \times n$ matrix of transformation T)

$\iff$ $rref(A) = I_n$ $\iff$ $rank(A) = n$

Lemma: Inverse of the linear transformation is linear.

Proof: Say $T^{-1}(\vec{x}) = \vec{u}$ and $T^{-1}(\vec{y}) = \vec{v}$, that is
$T(\vec{u}) = \vec{x}$ and $T(\vec{v}) = \vec{y}$. Now $T^{-1}(\vec{x} + \vec{y}) =$

$= T^{-1}(T(\vec{u}) + T(\vec{v})) \overset{\text{lin. } T}{=} T^{-1}(T(\vec{u} + \vec{v})) = \vec{u} + \vec{v} = T^{-1}(\vec{x}) + T^{-1}(\vec{y})$

$T(\vec{u}) = \vec{x}$ and $T(\vec{v}) = \vec{y}$. Now $T^{-1}(\vec{x}+\vec{y}) =$

$$= T^{-1}(T(\vec{u}) + T(\vec{v})) \overset{\text{lin.}T}{=} T^{-1}(T(\vec{u}+\vec{v})) = \vec{u}+\vec{v} = T^{-1}(\vec{x}) + T^{-1}(\vec{y})$$

$$T^{-1}(k\cdot\vec{x}) = T^{-1}(k\cdot T(\vec{u})) \overset{\text{lin.}T}{=} T^{-1}(T(k\cdot\vec{u})) = k\cdot\vec{u} = k\cdot T^{-1}(\vec{x}). \quad \square$$

Summing everything up,

Invertible matrices

A square matrix A is said to be *invertible* if the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case, the matrix of T^{-1} is denoted by A^{-1}. If the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

Invertibility

An $n \times n$ matrix A is invertible if (and only if)

$$\operatorname{rref}(A) = I_n$$

or, equivalently, if

$$\operatorname{rank}(A) = n.$$

Note, that I_n is the matrix of identity transformation $Id_{\mathbb{R}^n}: \mathbb{R}^n \to \mathbb{R}^n$. So, since $T \circ T^{-1} = T^{-1} \circ T = Id_{\mathbb{R}^n}$, then $AA^{-1} = A^{-1}A = I_n$ for $n \times n$ matrix A.

Now, consider the system $A\vec{x} = \vec{b}$. If A is invertible, $A(A^{-1}\vec{b}) = I_n\vec{b} = \vec{b}$, so $A^{-1}\vec{b}$ is a solution.

Invertibility and linear systems

Let A be an $n \times n$ matrix.

a. Consider a vector $\vec{b}$ in $\mathbb{R}^n$. If A is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If A is noninvertible, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions or none.

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b. Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If A is invertible, then this is the only solution. If A is noninvertible, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

Example: Is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$$

Invertible? We row reduce:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix} \begin{matrix} \\ -2(I) \\ -3(I) \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{pmatrix} \begin{matrix} -(II) \\ \\ -5(II) \end{matrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ \div -1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-(III)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 = \mathrm{rref}(A)$$

So, matrix A is invertible.

Lets find inverse of A, or equivalently, inverse of the linear transformation $\vec{y} = A\vec{x}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 2x_3 \end{pmatrix}$$

We solve the system for input variables x_1, x_2, x_3 to find how they are determined by y_1, y_2, y_3. Note that the procedure is the same as for our coding example in the previous lecture.

$$\left|\begin{array}{rcl} x_1 + x_2 + x_3 &=& y_1 \\ 2x_1 + 3x_2 + 2x_3 &=& y_2 \\ 3x_1 + 8x_2 + 2x_3 &=& y_3 \end{array}\right| \begin{array}{c} \longrightarrow \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{array}$$

$$\left|\begin{array}{rcl} x_1 + x_2 + x_3 &=& y_1 \\ x_2 &=& -2y_1 + y_2 \\ 5x_2 - x_3 &=& -3y_1 + y_3 \end{array}\right| \begin{array}{c} -(\mathrm{II}) \\ \longrightarrow \\ -5(\mathrm{II}) \end{array}$$

$$\left|\begin{array}{rcl} x_1 + x_3 &=& 3y_1 - y_2 \\ x_2 &=& -2y_1 + y_2 \\ -x_3 &=& 7y_1 - 5y_2 + y_3 \end{array}\right| \begin{array}{c} \longrightarrow \\ \\ \div(-1) \end{array}$$

$$\left|\begin{array}{rcl} x_1 + x_3 &=& 3y_1 - y_2 \\ x_2 &=& -2y_1 + y_2 \\ x_3 &=& -7y_1 + 5y_2 - y_3 \end{array}\right| \begin{array}{c} -(\mathrm{III}) \\ \longrightarrow \\ \\ \end{array}$$

$$\left|\begin{array}{rcl} x_1 &=& 10y_1 - 6y_2 + y_3 \\ x_2 &=& -2y_1 + y_2 \\ x_3 &=& -7y_1 + 5y_2 - y_3 \end{array}\right|.$$

So the inverse transformation is given by a matrix

$$A^{-1} = \begin{pmatrix} 10 & -6 & 1 \\ \dots \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}$$

Writing elimination above in a matrix form:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ -2(\mathrm{I}) \\ -3(\mathrm{I}) \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \begin{array}{c} -(\mathrm{II}) \\ \longrightarrow \\ -5(\mathrm{II}) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ \\ \div(-1) \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \begin{array}{c} -(\mathrm{III}) \\ \longrightarrow \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right].$$

Describing the process:

Finding the inverse of a matrix

To find the *inverse* of an $n \times n$ matrix A, form the $n \times (2n)$ matrix $\left[\, A \mid I_n \,\right]$ and compute rref $\left[\, A \mid I_n \,\right]$.

- If rref $\left[\, A \mid I_n \,\right]$ is of the form $\left[\, I_n \mid B \,\right]$, then A is invertible, and $A^{-1} = B$.
- If rref $\left[\, A \mid I_n \,\right]$ is of another form (i.e., its left half fails to be I_n), then A is not invertible. Note that the left half of rref $\left[\, A \mid I_n \,\right]$ is rref(A).

Operations on inverse matrices

The inverse of a product of matrices

If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}.$$

If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}.$$

Pay attention to the order of the matrices. (Order matters!)

Proof: exercise.

What follows is the useful result for finding matrix inverses.

A criterion for invertibility

Let A and B be two $n \times n$ matrices such that

$$BA = I_n. \quad \text{(Only one side suffices!)}$$

Then

a. A and B are both invertible,

b. $A^{-1} = B$ and $B^{-1} = A$, and

c. $AB = I_n$.

Proof: To demonstrate that A is invertible, it suffices to show that $A\vec{x} = 0$ has unique solution $\vec{x} = 0$. (Theorem above). $A\vec{x} = 0 \Rightarrow BA\vec{x} = B0 = 0 \Rightarrow$
$\Rightarrow I_n\vec{x} = 0 \Rightarrow \vec{x} = 0$ as claimed. Thus A is invertible.
$BA = I_n \Rightarrow BAA^{-1} = A^{-1} \Rightarrow B = A^{-1}$. B being inverse of A, is itself invertible, and $B^{-1} = (A^{-1})^{-1} = A$. Finally $AB = AA^{-1} = I_n$. $\square$

We can use this Theorem for ease of computation.
We claimed that

$$B = \begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \end{pmatrix} \text{ is the inverse of } A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$B=\begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix} \text{ is the inverse of } A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$$

$$BA=\begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I_3.$$

Lets discuss when is 2×2 matrix invertible. Say $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}=$$

$$=(ad-bc)\cdot I_2$$ if $ad-bc\neq 0$, we can write

$$\underbrace{\left(\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)}_{B}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}=I_2.$$

So A is invertible with inverse B. Conversly if A is invertible,

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}=(ad-bc)I_2$$ or equivalently

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=(ad-bc)A^{-1}$, since $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\neq 0$ we get that $(ad-bc)\neq 0$.

Inverse and determinant of a 2 × 2 matrix

a. The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if (and only if) $ad - bc \neq 0$.

Quantity $ad - bc$ is called the *determinant* of A, written $\det(A)$:

$$\det(A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

b. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$