Examples:

1) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{Represents}} \begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \\ 0 = 1 \end{cases}$

So, there are no solutions. This system is inconsistent

2) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{Represents}} \begin{cases} x_1 + 2x_2 = 1 \\ x_3 = 2 \end{cases}$

$\longrightarrow \begin{cases} x_1 = 1 - 2x_2 \\ x_3 = 2 \end{cases}$, assign $x_2 = t$

Solutions: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-2t \\ t \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$.

3) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \longrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$

Unique solution $\begin{pmatrix} x_1 \\ x_2 \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}$

Unique solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Number of solutions of a linear system

A system of equations is said to be *consistent* if there is at least one solution; it is *inconsistent* if there are no solutions.

A linear system is inconsistent if (and only if) the reduced row-echelon form of its augmented matrix contains the row $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$, representing the equation $0 = 1$.

If a linear system is consistent, then it has either

Recall geometric intuition

- *infinitely many solutions* (if there is at least one free variable), or
- *exactly one solution* (if all the variables are leading).

From the examples above, we see that number of leading 1's in the reduced row-echelon form tells us about the number of solutions of a linear system.

Definition: The rank of a matrix A is the number of leading 1's in $\mathrm{rref}(A)$, denoted $\mathrm{rank}(A)$.

Example:

$$\mathrm{rank}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 2, \text{ since } \mathrm{rref}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Observations: Consider a system with n equations and m variables, with $n \times m$ matrix A as a coefficient matrix.

a) $\mathrm{rank}(A) \leq n$ and $\mathrm{rank}(A) \leq m$

a) rank $(A) \leq n$ and rank$(A) \leq m$

By definition of rref, there is at most one leading 1 in each of the n rows and each of the m columns.

b) If the system is inconsistent, rank$(A) < n$.

rref of augmented matrix contains $[0\ 0\ 0 \cdots 0 \,|\, 1]$, so that rref$(A)$ contains $[0\ 0 \cdots 0]$, so rank$(A) < n$

c) System has exactly one solution, then rank$(A) = m$.

(*) $\begin{pmatrix} \text{number of} \\ \text{free variables} \end{pmatrix} = \begin{pmatrix} \text{total number} \\ \text{of variables} \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{leading var.} \end{pmatrix} =$

$$= m - \text{rank}(A)$$

No free variables $\Rightarrow$ $m = \text{rank}(A)$

d) If the system has infinitely many solutions, then rank$(A) < m$

Inf. many solutions $\Rightarrow$ at least one free variable

$\Rightarrow$ by (*), $m - \text{rank}(A) > 0$.

Number of equations vs. number of unknowns

a. If a linear system has exactly one solution, then there must be at least as many equations as there are variables ($m \leq n$ with the notation from Example 3 from above)

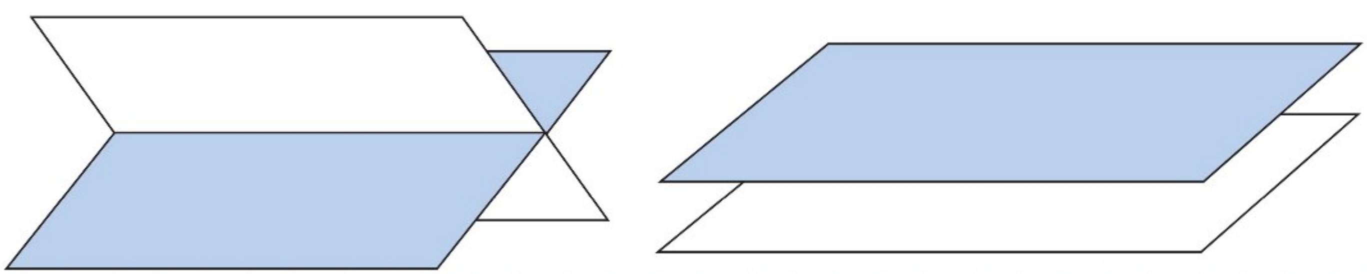
Equivalently, we can formulate the contrapositive:

b. A linear system with fewer equations than unknowns ($n < m$) has either no solutions or infinitely many solutions.

Proof: by a) and c) $m = \text{rank}(A) \leq n$. $\square$

Example: $\begin{cases} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = a_{14} \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = a_{24} \end{cases}$ gives either two parallel planes or planes intersecting in a line.

But they can never intersect at a point!



Question: When does the system with n equations and n variables have exactly one solution?

System has exactly one solution $\Rightarrow$ $\text{rank}(A) = m = n$.

Conversely, if $\text{rank}(A) = n$, then

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and this system has exactly one solution.

So,

Systems of n equations in n variables

A linear system of n equations in n variables has a unique solution if (and only if) the rank of its coefficient matrix A is n. In this case,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else.

Some Matrix operations

From school you are aware of some operations on vectors

Example: Addition of vectors in $\mathbb{R}^n$.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \\ -7 \end{pmatrix}, \quad \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$$

Scalar multiplication:

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}, \quad 3.5 \cdot \vec{w}_1 = 3.5 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 7 \\ 10.5 \\ 14 \end{pmatrix}$$

$$\vec{w}_1 = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}, \quad 3.5 \cdot w_1 = 3.5 \cdot \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 10.5 \\ 14 \\ 21 \end{pmatrix}$$

As for vectors, sums and scalar multiples of vectors are defined enty by entry

Sums:

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$

Scalar multiples:

$$k \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} k \cdot a_{11} & \cdots & k a_{1m} \\ \vdots & & \vdots \\ k a_{n1} & \cdots & k a_{nm} \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 3 & 1 \\ 5 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 4 \\ 9 & 8 & 5 \end{pmatrix}$$

$$3 \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -3 & 9 \end{pmatrix}$$

From school, you also know about dot product of vectors. Lets recall

Dot product of vectors

Consider two vectors $\vec{v}$ and $\vec{w}$ with components $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, respectively. Here $\vec{v}$ and $\vec{w}$ may be column or row vectors, and the two vectors need not be of the same type. The dot product of $\vec{v}$ and $\vec{w}$ is defined to be the scalar

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n.$$

Note that our definition of the dot product isn't row-column-sensitive. The dot product does not distinguish between row and column vectors.

Example: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 = 11$

Now, we define a product of a matrix and a vector

The product $A\vec{x}$

If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \ldots, \vec{w}_n$, and $\vec{x}$ is a vector in $\mathbb{R}^m$, then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

In words, the ith component of $A\vec{x}$ is the dot product of the ith row of A with $\vec{x}$.

Note that $A\vec{x}$ is a column vector with n components, that is, a vector in $\mathbb{R}^n$.

Example: 1) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \end{pmatrix} =$

$$= \begin{pmatrix} 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 - 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \\ 1\cdot 3 + 0\cdot 1 - 1\cdot 2 \end{pmatrix} = \begin{pmatrix} \\ 1 \end{pmatrix}$$

2) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 \\ 0 x_1 + 1\cdot x_2 + 0\cdot x_3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

for any $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

Note that, $A\vec{x}$ only defined when number of columns of A matches the number of components of vector $\vec{x}$:

$$\underbrace{\overbrace{A}^{n\times m} \cdot \overbrace{\vec{x}}^{m\times 1}}_{n\times 1}$$

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ – undefined!

There is another, equivalent way to look at matrix-vector multiplication. Lets look at an example:

$$A\vec{x} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2\cdot 1 + 3\cdot 2 \\ 1\cdot 3 + 0\cdot 1 + -1\cdot 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot 3 \\ 1\cdot 3 \end{pmatrix} + \begin{pmatrix} 2\cdot 1 \\ 0\cdot 1 \end{pmatrix} + \begin{pmatrix} 3\cdot 2 \\ -1\cdot 2 \end{pmatrix} =$$

$$= 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} =$$

$$= 3\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}2\\0\end{pmatrix} + 2\begin{pmatrix}3\\-1\end{pmatrix} =$$

$$= x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + x_3 \cdot v_3$$

where $A = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

So, we get

The product $A\vec{x}$ in terms of the columns of A

If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x}$ is a vector in $\mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Proof:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{pmatrix}$$

$$(i^{th} \text{ component of } A\vec{x}) \overset{\text{by def}}{=} \vec{w}_i \cdot X = a_{i1}x_1 + \cdots + a_{im}x_m$$

$$= x_1(i^{th} \text{ component of } \vec{v}_1) + \cdots + x_m(i^{th} \text{ component of } \vec{v}_m)$$

$$= i^{th} \text{ component of } (x_1 \cdot \vec{v}_1 + \cdots + x_m\vec{v}_m) \qquad \square$$

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix}1\\4\\7\end{pmatrix} + (-4)\begin{pmatrix}2\\5\\8\end{pmatrix} + 2\begin{pmatrix}3\\6\\9\end{pmatrix} =$

Note „zero

$$= \begin{pmatrix}2\\8\\14\end{pmatrix} - \begin{pmatrix}8\\20\\32\end{pmatrix} + \begin{pmatrix}6\\12\\8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Note „zero devisor" kind of thing happening here.

Such expressions like $\sum_{i=1}^{m} x_i \vec{v}_i = x_1\vec{v}_1 + \cdots + x_m \vec{v}_m$ are central to linear algebra.

Linear combinations

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Example: is $\begin{pmatrix}1\\1\\1\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\\6\end{pmatrix}$?

So, we want x, y, such that $\begin{pmatrix}1\\1\\1\end{pmatrix} = x\begin{pmatrix}1\\2\\3\end{pmatrix} + y\begin{pmatrix}4\\5\\6\end{pmatrix} =$

$= \begin{pmatrix}x+4y\\2x+5y\\3x+6y\end{pmatrix}$. So we need to solve

$$\begin{cases} x+4y=1\\ 2x+5y=1\\ 3x+6y=1\end{cases}. \quad \mathrm{rref}\left(\begin{array}{cc|c}1&4&1\\2&5&1\\3&6&1\end{array}\right) = \left(\begin{array}{cc|c}1&0&-\frac{1}{3}\\0&1&\frac{1}{3}\\0&0&0\end{array}\right)$$

So, $x = -\frac{1}{3}$ and $y = \frac{1}{3}$. That is

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = -\frac{1}{3}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{1}{3}\cdot\begin{pmatrix}4\\5\\6\end{pmatrix}$$

As we will see later, next is a very important property of Matrices:

Algebraic rules for $A\vec{x}$

If A is an $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^m$, and k is a scalar, then

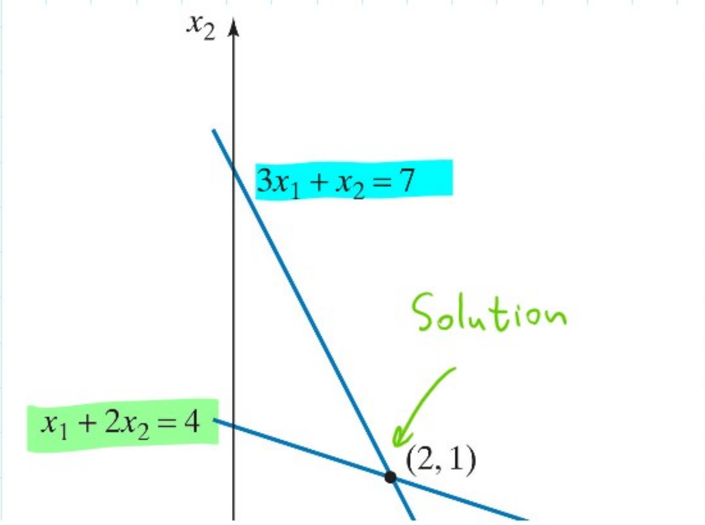
a. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$, and

b. $A(k\vec{x}) = k(A\vec{x})$.

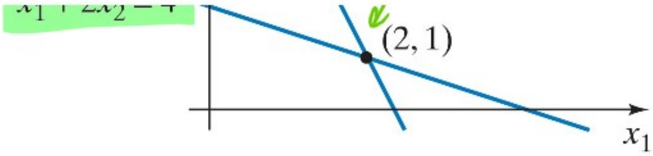
Proof: On central exercise worksheet $\square$.

Now, we can look at linear systems from a new perspective. Consider:

$$\begin{cases} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right)$$



Solution can be represented geometrically as an intersection of two lines $3x_1 + x_2 = 7$ and $x_1 + 2x_2 = 4$.



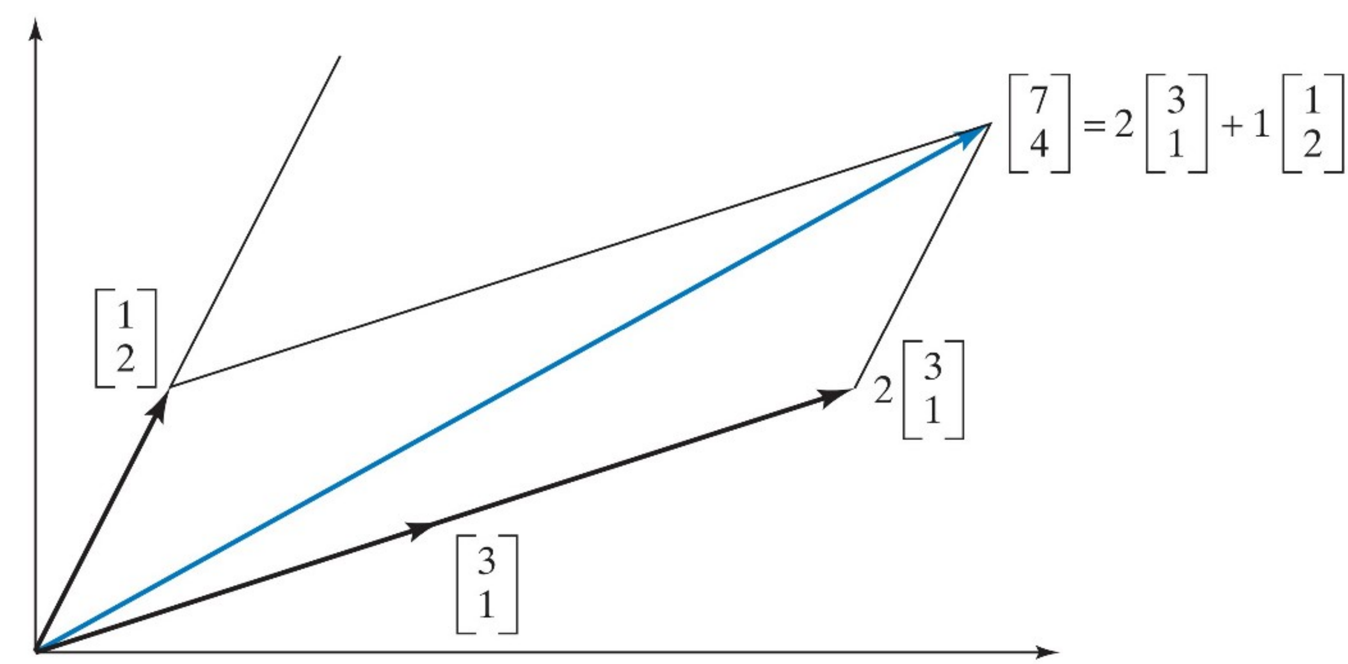
Alternatively, we can write

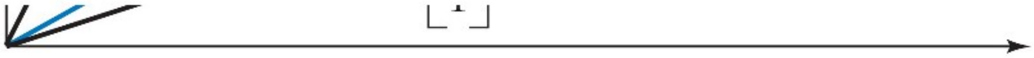
$$\begin{pmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 3x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \text{ or}$$

$$x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix},$$

So, solving this system amounts to writing the vector $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix},$$ solution geometrically





We can also go further and write $x_1\begin{pmatrix}3\\1\end{pmatrix}+x_2\begin{pmatrix}1\\2\end{pmatrix}$ as

$\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$, and linear system now takes form $\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$

This is called a matrix form of a linear system.

$$\begin{cases}3x_1+x_2=7\\x_1+2x_2=4\end{cases}\longrightarrow\left(\begin{array}{cc|c}3 & 1 & 7\\1 & 2 & 4\end{array}\right)\longrightarrow\underbrace{\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x_1\\x_2\end{pmatrix}}_{\vec{x}}=\underbrace{\begin{pmatrix}7\\4\end{pmatrix}}_{\vec{b}}$$

So, we can write $A\vec{x}=\vec{b}$

Matrix form of a linear system

We can write the linear system with augmented matrix $[A \mid \vec{b}]$ in matrix form as

$$A\vec{x}=\vec{b}.$$

Remark. The i^{th} component of $A\vec{x}=\vec{b}$ equation is the i^{th} equation $a_{i1}x_1+\dots+a_{im}x_m=b_i$ of the system with augmented matrix $(A \mid \vec{b})$.

Solving the linear system $A\vec{x}=\vec{b}$ amounts to expressing $\vec{b}$ as a linear combination of the column vectors of A.

vectors of A.

Example:
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 1 \\ x_1 + 4x_2 \qquad\quad = 0 \end{cases}$$

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 1 & 4 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 3 & -4 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ or $$A\vec{x} = \vec{b}.$$