Lecture 12 - Eigenvalues and Eigenvectors I

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We already encounted the fact that it is much easier to deal with diagonal matrices; for example, it is easier to find powers, rank and determinant of a diagonal matrix.

For this lecture, we only consider finite dimentional vector spaces over $\mathbb{R}$.

Definition: Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$. Then $T(\vec{x}) = A\vec{x}$ for some $n \times n$ matrix A. We say that A (or T) is diagonalizable, if there exists a basis of $\mathbb{R}^n$ such that matrix of T in this basis is diagonal.

Now, we know that if S is a base change matrix then matrix A in new basis is written as $S^{-1}AS =: B$, and we say that A and B are similar. So to diagonalize the square matrix A, we need to find a diagonal matrix B, such that $S^{-1}AS = B$ for some invertible matrix S.

For a basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$, we also know that

$$B := [A]_{\mathcal{B}} = \begin{pmatrix} | & | & & | \\ [T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \\ | & | & & | \end{pmatrix}$$

So, $[A]_{\mathcal{B}}$ is diagonal if and only if $[T(v_i)]_{\mathcal{B}} = \lambda_i [v_i]_{\mathcal{B}}$,

for some n numbers $\lambda_i \in \mathbb{R}$. Now $[T(v_i)]_{\mathcal{B}} = \lambda_i [v_i]$

So, $[A]_{\mathcal{B}}$ is diagonal if and only if $[T(v_i)]_{\mathcal{B}} = \lambda_i [e_i]_{\mathcal{B}}$,
for some n numbers $\lambda_i \in \mathbb{R}$. Now, $[T(v_i)]_{\mathcal{B}} = \lambda_i [v_i]_{\mathcal{B}}$ if and only if for any other basis $\mathcal{C}$ and basis change matrix $S_{\mathcal{C}}$, $[T(v_i)]_{\mathcal{C}} = S_{\mathcal{C}} \cdot \lambda_i [v_i]_{\mathcal{B}} = \lambda_i [v_i]_{\mathcal{C}}$, in particular, if and only if $Av_i = \lambda_i v_i$, $i = 1, \dots, n$. So,

$$[A]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$$

and we conclude that $n \times n$ matrix A is diagonalizable if and only if there exists a basis $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ such that $A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$ for some numbers $\lambda_i \in \mathbb{R}$.

Eigenvectors, eigenvalues, and eigenbases[1]

Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$.

A nonzero vector $\vec{v}$ in $\mathbb{R}^n$ is called an **eigenvector** of A (or T) if

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ. This λ is called the **eigenvalue** associated with eigenvector $\vec{v}$.

A basis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$ is called an **eigenbasis** for A (or T) if the vectors $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors of A, meaning that $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$ for some scalars $\lambda_1, \dots, \lambda_n$.

So, $v \in \mathbb{R}^n$ is an eigenvector of A, if $Av \in \mathrm{span}(v)$.

And we demonstrated:

Theorem: Matrix A is diagonalizable if and only if there exists an eigenbasis of A. // $\vec{v}_1, \dots, \vec{v}_n$ is an eigenbasis

Theorem: Matrix A is diagonalizable if and only if there exists an eigenbasis of A. If $\vec{v}_1, \ldots, \vec{v}_n$ is an eigenbasis of A, with $A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, A\vec{v}_n = \lambda_n \vec{v}_n$, then matrices

$$S = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

diagonalize A, meaning $S^{-1}AS = B$. Conversly, if the matrices S and B diagonalize A, then column vectors of S form the eigenbasis of A, and the diagonal entries of B are corresponding eigenvalues.

We can also prove this result directly: Suppose there exists an eigenbasis $(v_1, \ldots, v_n)$ of A, then

$$AS = A\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix} =$$

$$= \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} = SB$$

Conversly, if $S^{-1}AS = B$ for invertible S and diagonal B then $AS = SB$ and

$$AS = A\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =$$

$$AS = A\begin{pmatrix} v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\begin{pmatrix} \ddots & \\ 0 & \lambda_n \end{pmatrix} =$$

$$= \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix}, \text{ so } \quad Av_i = \lambda_i v_i, \quad i = 1, \dots, n.$$

Say now 0 is an eigenvalue of A. Then there exists a nonzero vector $\vec{v} \in \mathbb{R}^n$, such that $A\vec{v} = 0 \cdot \vec{v} = 0$, therefore $v \in \ker A \neq \{0\}$. So A is not injective, thus not invertible. So, we add:

Various characterizations of invertible matrices

For an $n \times n$ matrix A, the following statements are equivalent.

- **i.** A is invertible.
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in $\mathbb{R}^n$.
- **iii.** rref $A = I_n$.
- **iv.** rank $A = n$.
- **v.** im $A = \mathbb{R}^n$.
- **vi.** ker $A = \{\vec{0}\}$.
- **vii.** The column vectors of A form a basis of $\mathbb{R}^n$.
- **viii.** The column vectors of A span $\mathbb{R}^n$.
- **ix.** The column vectors of A are linearly independent.
- **x.** det $A \neq 0$.
- **xi.** 0 fails to be an eigenvalue of A.

Interlude to dynamical systems

Norwestern Mexican desert is populated by mainly two spicies of animals: coyotes and roadrunners. We wish to model populations of $c(t)$ and $r(t)$ of

We wish to model populations of $c(t)$ and $r(t)$ of coyotes and roadrunners respectively t years from now if current populations c_0 and r_0 are known. (This is a simplified model)

It is known that

$$c(t+1) = 0.86\,c(t) + 0.08\,r(t)$$
$$r(t+1) = -0.12\,c(t) + 1.14\,r(t)$$

where $t+1$ is the next year after t. The reason for the coefficent $0.86 < 1$ is because number of coyotes this year should decrease in the absense of pray (roadrunners) and situation is opposite for roadrunners, (hence $1.14 > 1$). Also, the number of roadrunners positivly effects number of coyotes $(0,08 > 0)$ and opposite for roadrunners $(-0.12 < 0)$. In matrix notation

$$\vec{x}(t+1) := \begin{pmatrix} c(t+1) \\ r(t+1) \end{pmatrix} = \begin{pmatrix} 0.86\,c(t) + 0.08\,r(t) \\ -0.12\,c(t) + 1.14\,r(t) \end{pmatrix} = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} c(t) \\ r(t) \end{pmatrix}$$

If $A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}$, then $\vec{x}(t+1) = A\,\vec{x}(t)$.

So, the transformation that system undergoes in

So, the transformation that system undergoes in one year is linear $\vec{x}(t) \xrightarrow{A} \vec{x}(t+1)$. Say we know the inital state $\vec{x}(0) = \begin{pmatrix} c_0 \\ r_0 \end{pmatrix}$, then we can find

$$\vec{x}(t) = A \cdot \vec{x}(t-1) = A \cdot A \, \vec{x}(t-2) = \cdots = \underbrace{A \cdot A \cdots A}_{t \text{ times}} \vec{x}(0) = A^t \vec{x}(0)$$

For example, if $\vec{x}(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$, in ten years we will have

```
A = matrix([[0.86,  0.08],
            [-0.12, 1.14]])
x0 = vector([100,100])
x10= A^10*x0
```

```
show(x10.column())
```

$$\begin{pmatrix} 79.7691244099999 \\ 169.571685210000 \end{pmatrix}$$

So, rounding up

$$\vec{x}(10) \approx \begin{pmatrix} 80 \\ 170 \end{pmatrix}.$$

So 80 coyotes and 170 roadrunners after 10 years.

To better understand the long term behaviour of the system it would be usefull to have closed formulas for $c(t)$ and $r(t)$. For this we will go beyond numerical methods.

Case 1) Suppose $c_0 = 100$ and $r_0 = 300$. Then

```
x0 = vector([100,300])
x1= A*x0
show('x(1) =',x1.column())
```

$$x(1) = \begin{pmatrix} 110 \\ 330 \end{pmatrix}$$

$$\vec{x}(1) = A \cdot \vec{x}(0) = \begin{pmatrix} 110 \\ 330 \end{pmatrix} = 1.1 \begin{pmatrix} 100 \\ 300 \end{pmatrix} = 1.1\, \vec{x}(0)$$

So, the vector $\begin{pmatrix} 100 \\ 300 \end{pmatrix}$ is an eigenvector of A with corresponding eigenvalue 1.1. In particular,

$$\vec{x}(t) = A^t \vec{x}(0) = A^{t-1}\, 1.1\, \vec{x}(0) = 1.1\, A^{t-1} x(0) = (1.1)^2 A^{t-2} \cdot \vec{x}(0) = \cdots$$

$$\cdots = (1.1)^t \vec{x}(0),$$ and we also conclude that $(1.1)^t$ is an eigenvalue of A^t (This is true for eigenvalues in general, exercise in CE)

So, in particular

$$c(t) = (1.1)^t \cdot 100 \quad \text{and} \quad r(t) = (1.1)^t\, 300$$

and both populations will grow exponentially, by 10% each year.

Case 2) Suppose $c_0 = 200$, $r_0 = 100$. Then

$$\vec{x}(1) = A\vec{x}(0) = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 180 \\ 90 \end{pmatrix} = 0.9 \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

So, $\vec{x}(0) = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$ is another eigenvector of A with eigenvalue 0.9. So, similarly

$$\vec{x}(t) = A^t \vec{x}(0) = (0.9)^t \vec{x}(0) = \begin{pmatrix} 0.9^t \cdot 200 \\ \end{pmatrix}$$

$$\vec{x}(t) = A^t \vec{x}(0) = (0.9)^t \vec{x}(0) = \begin{pmatrix} 0.9^t \cdot 200 \\ 0.9^t \cdot 100 \end{pmatrix}$$

So, $c(t) = (0.9)^t \, 200$ and $r(t) = (0.9)^t \cdot 100$

both populations are decreasing by 10% each year: too many coyotes are chasing too few roadrunners, a bad state of affairs for both species.

case 3) Say $c_0 = r_0 = 1000$. Then,

```
A = matrix(QQ, [[0.86,  0.08], #QQ stands for rational numbers
                [-0.12, 1.14]])
x0 = vector([1000,1000])
x1= A*x0
```

```
show(x1.column())
```

$$\begin{pmatrix} 940 \\ 1020 \end{pmatrix}$$

and vector $\begin{pmatrix} 940 \\ 1020 \end{pmatrix}$ fails to be an eigenvector of A, so we can not directly find the closed formula.

The idea is to work with eigenbasis

$$\vec{v}_1 = \begin{pmatrix} 100 \\ 300 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

considered above. Any vector $\vec{x} \in \mathbb{R}^2$ can be written as linear combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$.

In particular, $\vec{x}(0) = 2\vec{v} + 4\vec{v}$,

$$\begin{pmatrix} 1000 \end{pmatrix} = 2 \cdot \begin{pmatrix} 100 \end{pmatrix} + 4 \begin{pmatrix} 200 \end{pmatrix}$$

$$\begin{pmatrix}1000\\1000\end{pmatrix} = 2\cdot\begin{pmatrix}100\\300\end{pmatrix} + 4\begin{pmatrix}200\\100\end{pmatrix}$$

Therefore, in this case,

$$\begin{aligned}\vec{x}(t) &= A^t\vec{x}(0) = A^t(2\cdot\vec{v}_1 + 4\vec{v}_2) = 2A^t\vec{v}_1 + \\ &+ 4A^t\vec{v}_2 = 2(1.1)^t\vec{v}_1 + 4(0.9)^t\vec{v}_2. \\ &= 2(1.1)^t\begin{pmatrix}100\\300\end{pmatrix} + 4(0.9)^t\begin{pmatrix}200\\100\end{pmatrix}\end{aligned}$$

Therefore,

$$c(t) = (1.1)^t\cdot 200 + (0.9)^t\cdot 800$$
$$r(t) = (1.1)^t\cdot 600 + (0.9)^t\cdot 400$$

(as $t\to\infty$, each population grows by 10% and $r(t)/c(t) \to 600/200 = 3$)

The derivation of the formula above is equivalent to diagonalization:

We take $S = \begin{pmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{pmatrix} = \begin{pmatrix}100 & 200\\ 300 & 100\end{pmatrix}$, then

$$S^{-1}AS = B = \begin{pmatrix}1.1 & 0\\ 0 & 0.9\end{pmatrix}, \text{ thus } A = SBS^{-1} \text{ and}$$

$$A^t = (SBS^{-1})^t = \underbrace{SBS^{-1}SBS^{-1}\cdots SBS^{-1}}_{t \text{ times}} = SB^tS^{-1}$$

t times

Now, $\vec{x}(t) = SB^tS^{-1}\vec{x}(0)$, and using Python:

```
x0 = vector([1000,1000])
S = matrix([[100, 200],
            [300, 100]])

show(S^(-1)*x0.column())
```

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Thus $S^{-1}\vec{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ – a coordinate vector of the initial state vector with respect to a given eigenbasis.

Again, using python

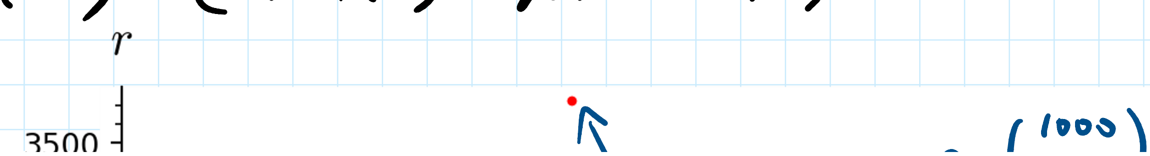
$$\vec{x}(t) = SB^tS^{-1}\vec{x}(0) = \begin{pmatrix} 100 & 200 \\ 300 & 100 \end{pmatrix}\begin{pmatrix} (1.1)^t & 0 \\ 0 & (0.9)^t \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} =$$

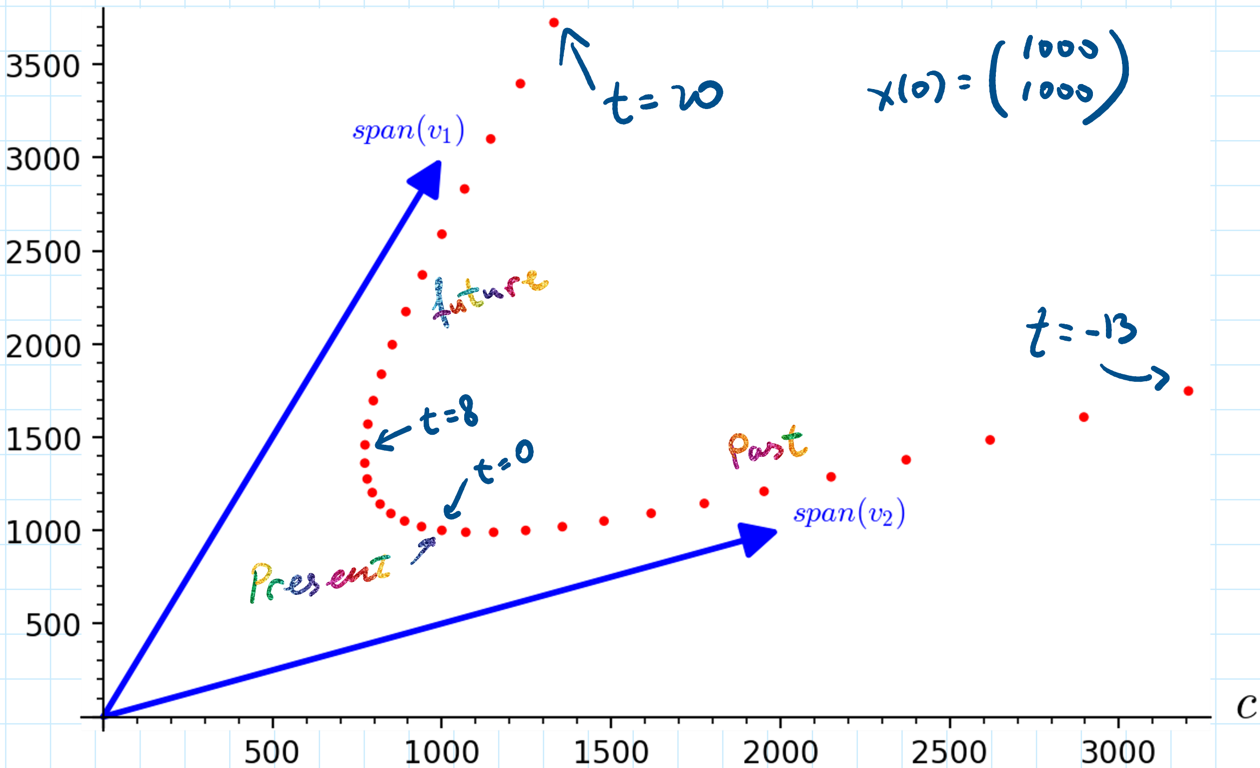
$$= \begin{pmatrix} 100 & 200 \\ 300 & 100 \end{pmatrix}\begin{pmatrix} 2(1.1)^t \\ 4(0.9)^t \end{pmatrix} = 2\cdot(1.1)^t\begin{pmatrix} 100 \\ 300 \end{pmatrix} + 4(0.9)^t\begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

Which is what we expected.

To show the evolution of out coyote-roadrunner system, we would like to plot vectors $\vec{x}(t)$ as points for different t. We will also consider negative t, to model what happened to the system in the past $(t = -1, -2, -3, \dots)$. Note that $\vec{x}(0) = A\vec{x}(-1)$, so that $\vec{x}(-1) = A^{-1}\vec{x}(0)$, if A is invertible (as in our example). Likewise

$\vec{x}(-t) = (A^t)^{-1}\vec{x}(0)$ for $t = 2, 3, \dots$






```
system = point2d([((A^n*x0)[0],(A^n*x0)[1]) for n in IntegerRange(-13,20)], axes_labels=['$c$', '$r$'], \
            color ='red', size=10,dpi=200)
v1 = plot(vector([1000,3000]))+text('$span(v_1)$', (9*100, 10.5*300)) #span of v1
v2 = plot(vector([2000,1000]))+text('$span(v_2)$', (11*200, 11*100)) #span of v2
system+v1+v2
```

To get a sense of long term behaviour, we can also sketch number of different trajectories by changing initial state $x(0)$. For example, this will include trajectories moving along lines $span(v_1)$ and $span(v_2)$, respectively with initial values $w_1 \in span(v_1)$ and $w_2 \in span(v_2)$, also a trajectory given by red dots on our figure from initial value $\begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$.

After sketching, we see that population will prosper in the long term if the ratio $\frac{c_0}{r_0}$ of the initial population exceed $1/2$; otherwise both populations population exceed 1/2; otherwise both populations will die out: