Lecture 10 - Linear Transformations of Vector spaces

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Consider two vector spaces V, W (over the same field $\mathbb{F}$). A function $T: V \to W$ is called a linear map (transformation) if

i) $T(v+w) = T(v) + T(w)$ for all $v, w \in V$

ii) $T(k \cdot v) = k \cdot T(v)$ for all $v \in V$ and $k \in \mathbb{F}$

Of course, then we define

$$Im(T) = \{ u \in W \mid \exists v \in V,\ T(v) = u \}$$
$$= \{ T(v) \mid v \in V \}$$

and

$$Ker(T) = \{ v \in V \mid T(v) = 0 \}$$

That is, $Ker(T) \subseteq V$ and $Im(T) \subseteq W$ are subspaces. (Check this.)

If the space $Im(T)$ is finite dimensional, then $\dim(Im(T))$ is called **rank** of T, and if $Ker(T)$ is finite dimensional, $\dim(Ker(T))$ is called **nullity** of T.

Theorem: Say V, W are vector spaces and V is finite dimensional. Let $T: V \to W$ be a linear map. Then:

$$\dim(V) = rank(T) + nullity(T) = \dim(Im(T)) + \dim(Ker(T)).$$

Proof: Exactly the same as for $\mathbb{R}^n$. Exercise!

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Warning: This only works (and makes sense) for finite dimensional V.

Example: 1) Left translation: let V be a space of all infinite sequences of, say, complex numbers (we can also take sequences of any other field, like $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}/2$ etc.). Consider the map

$$L : V \longrightarrow V$$
$$(z_0, z_1, z_2, \dots) \longmapsto (z_1, z_2, z_3, z_4, \dots)$$

L is a linear transformation:

$$L((z_0, z_1, \dots) + (w_0, w_1, \dots)) = L((z_0 + w_0, z_1 + w_1, \dots)) =$$
$$= (z_1 + w_1, z_2 + w_2, \dots) = L((z_0, z_1, \dots)) + L((w_0, w_1, \dots))$$
$$L(\lambda(z_0, z_1, z_2, \dots)) = L((\lambda z_0, \lambda z_1, \dots)) = (\lambda z_1, \lambda z_2, \dots)$$
$$= \lambda L((z_0, z_1, \dots))$$

$\operatorname{Ker} L$:

$$L((z_0, z_1, \dots)) = 0_V = (0, 0, \dots) \Rightarrow (z_1, z_2, z_3, \dots) = (0, 0, \dots)$$
$$\Rightarrow z_i = 0 \text{ for } i = 1, 2, \dots \text{. Therefore}$$

$$\operatorname{Ker} L = \{ (z_0, 0, 0, 0, \dots) \in V \mid z_0 \in \mathbb{C} \}$$ and

$\operatorname{nullity}(L) = 1$, since $(1, 0, 0, \dots)$ is a basis of $\operatorname{Ker} L$.

$\operatorname{Im} L$: For any sequence $(z_0, z_1, z_2, \dots)$ we have that, for example, $L((0, z_0, z_1, z_2, \dots)) = (z_0, z_1, z_2, \dots)$

that, for example, $L((0, z_0, z_1, z_2, \dots)) = (z_0, z_1, z_2, \dots)$ therefore any sequence is in the image of L. So, $\mathrm{Im}(L) = V$

2) Let $C[0,1]$ denote the vector space of all continuous functions from closed interval $[0,1]$ to real numbers $\mathbb{R}$. Define a map

$$I : C[0,1] \longrightarrow \mathbb{R}$$
$$f \longmapsto \int_0^1 f(x)\,dx$$

I is linear:

$$I(f+g) = \int_0^1 (f(x)+g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = I(f) + I(g)$$
$$I(r \cdot f) = \int_0^1 r f(x)\,dx = r \int_0^1 f(x)\,dx = r \cdot I(f).$$

$\mathrm{Im}\, I$:

$b \in \mathrm{Im}(I)$ if and only if there exists a function $f \in C[0,1]$, such that $\int_0^1 f(x)\,dx = b$. One may possibly just choose $f(x) = b$, therefore $\mathrm{Im}\, I = \mathbb{R}$.

$\ker I$:

$f \in \ker I \iff \int_0^1 f(x)\,dx = 0$, So $\ker I = \{ f \in C[0,1] \mid \int_0^1 f(x)\,dx = 0 \}$

3) Let $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \xrightarrow{T} \mathbb{R}^4$ be given by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

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$$T\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix} + \begin{pmatrix} e & f \\ g & h\end{pmatrix}\right) = T\begin{pmatrix} a+e & b+f \\ c+g & d+h\end{pmatrix} = \begin{pmatrix} a+e \\ b+f \\ c+g \\ d+h\end{pmatrix} = T\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) + T\left(\begin{pmatrix} e & f \\ g & h\end{pmatrix}\right),$$

$$T\left(r \cdot \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = T\left(\begin{pmatrix} ra & rb \\ rc & rd\end{pmatrix}\right) = \begin{pmatrix} ra \\ rb \\ rc \\ rd\end{pmatrix} = rT\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$$

Im(T):

Say $b = \begin{pmatrix} x \\ y \\ z \\ u\end{pmatrix} \in \mathbb{R}^4$, then $T\left(\begin{pmatrix} x & y \\ z & u\end{pmatrix}\right) = b$, therfor $\operatorname{Im} T = \mathbb{R}^4$.

Ker(T):

Say $T\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0\end{pmatrix} \iff a=b=c=d=0 \Rightarrow \begin{pmatrix} a & b \\ c & d\end{pmatrix} = 0$

So, $\operatorname{Ker} T = \{0\}$.

4) Let V be a finite dimensional vector space over $\mathbb{F}$ and $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V. Define a map

$$[\ ]_{\mathcal{B}} : V \to \mathbb{F}^n$$

by $[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n\end{pmatrix}$, where $v = c_1 v_1 + \dots + c_n v_n$ for unique $c_1, \dots, c_n \in \mathbb{F}$.

This map is called „coordinate transformation" or „a basis choice".

$[\ ]_{\mathcal{B}}$ is linear: Say $v = \sum c_i v_i$ and $w = \sum d_i v_i$

then, $[v+w]_{\mathcal{B}} = [\sum c_i v_i + \sum d_i v_i]_{\mathcal{B}} =$

$$= [\sum (c_i+d_i) v_i]_{\mathcal{B}} = \begin{pmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n\end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n\end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n\end{pmatrix} = [v]_{\mathcal{B}} + [w]_{\mathcal{B}}$$

$= [\sum (c_i+d_i) v_i]_{\mathcal{B}} = \begin{pmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = [v]_{\mathcal{B}} + [w]_{\mathcal{B}}$

$$[\lambda \cdot v]_{\mathcal{B}} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda [v]_{\mathcal{B}}$$

$Im([\ ]_{\mathcal{B}})$: for any $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}^n$, we have $[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ for a some $v = \sum_{i=1}^{n} c_i v_i$, and since every vector is uniquely written as a sum of basis vectors, $Im([\ ]_{\mathcal{B}}) = \mathbb{F}^n$ and $[\ ]_{\mathcal{B}}$ is a bijection.

Def: Linear transformation that is a bijection is called an isomorphism of vector spaces.

Say $T: V \to W$ is an isomorphism of vector spaces. Since T is a bijection there exists an inverse map $T^{-1}: W \to V$.

Lemma: The inverse map of a linear transformation is linear.

Proof: Exactly the same as for $\mathbb{R}^n$. (Exercise)

Isomorphic vector spaces are the „same" for all intents and purposes. Example 3) above showed that any fin. dimensional vector space is isomorphic to $\mathbb{F}^n$, so we do not really need new theory to study general finite dimensional vector spaces (theory that studies n-tuples $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ suffices). However study that studies n-tuples $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ suffices). However study of infinite dimensional vector spaces is different! (Functional analysis).

Remark: example 2) above is also just a coordinate transformation with respect to basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ of } Mat_{2\times 2}(\mathbb{R}).$$

So, as vector spaces over $\mathbb{R}$, $Mat_{2\times 2}(\mathbb{R})$ is the „same" as $\mathbb{R}^4$ (Note that as vector space, $Mat_{2\times 2}(\mathbb{R})$ does not take multiplication of matrices into account).

Properties of isomorphisms

a. A linear transformation T from V to W is an isomorphism if (and only if) $\ker(T) = \{0\}$ and $\operatorname{im}(T) = W$.

In parts (b) through (d), the linear spaces V and W are assumed to be finite dimensional.

b. The linear space V is isomorphic to W if (and only if) $\dim(V) = \dim(W)$.

c. Suppose T is a linear transformation from V to W with $\ker(T) = \{0\}$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

d. Suppose T is a linear transformation from V to W with $\operatorname{im}(T) = W$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

Proof: a) Say $T: V \to W$ is an isomorphism. Then T in particular is an injection, so if $T(v) = 0 = T(0)$ then $v = 0$. Therefore $\ker T = \{0\}$; also, T is then $v=0$. Therefore $\ker T = \{0\}$; also, T is surjective, and that $\operatorname{Im} T = W$.

Say, $\ker T = \{0\}$ and $\operatorname{Im} T = W$. Since $\operatorname{Im} T = W$ T is surjective. if $T(v) = T(w)$ then $T(v) - T(w) = 0$
$\Rightarrow T(v-w) = 0 \Rightarrow$ by $\ker T = \{0\}$, $v - w = 0$
$\Rightarrow v = w$. Thus T is an isomorphism.

b) Say $T: V \to W$ is an isomorphism. By rank-nullity

$$\dim V = \dim(\ker T) + \dim(\operatorname{Im} T) = 0 + \dim W = \dim W$$

Conversly, say $\dim V = \dim W = n$. Then V is isomorphic to $\mathbb{F}^n$ and W is also isomorpic to $\mathbb{F}^n$. Say respective isomorphisms are L and T.

$L: V \to \mathbb{F}^n$ and $T: W \to \mathbb{F}^n$. But then

$V \xrightarrow{L} \mathbb{F}^n \xrightarrow{T^{-1}} W$ is an isomorphism from V to W. (Composition of isomorphisms is isomorphism in CE).

c) We showed that T is injective. We have to show that $\operatorname{Im}(T) = W$ or equivalently, $\dim(\operatorname{Im} T) = W$. by rank-nullity for $T: V \to W$

$$\dim(W) = \dim(V) = \dim \ker T + \dim \operatorname{Im} T = \dim \operatorname{Im} T$$

d) By a) it suffices to show $\ker T = \{0\}$. We conclude by rank-nullity.

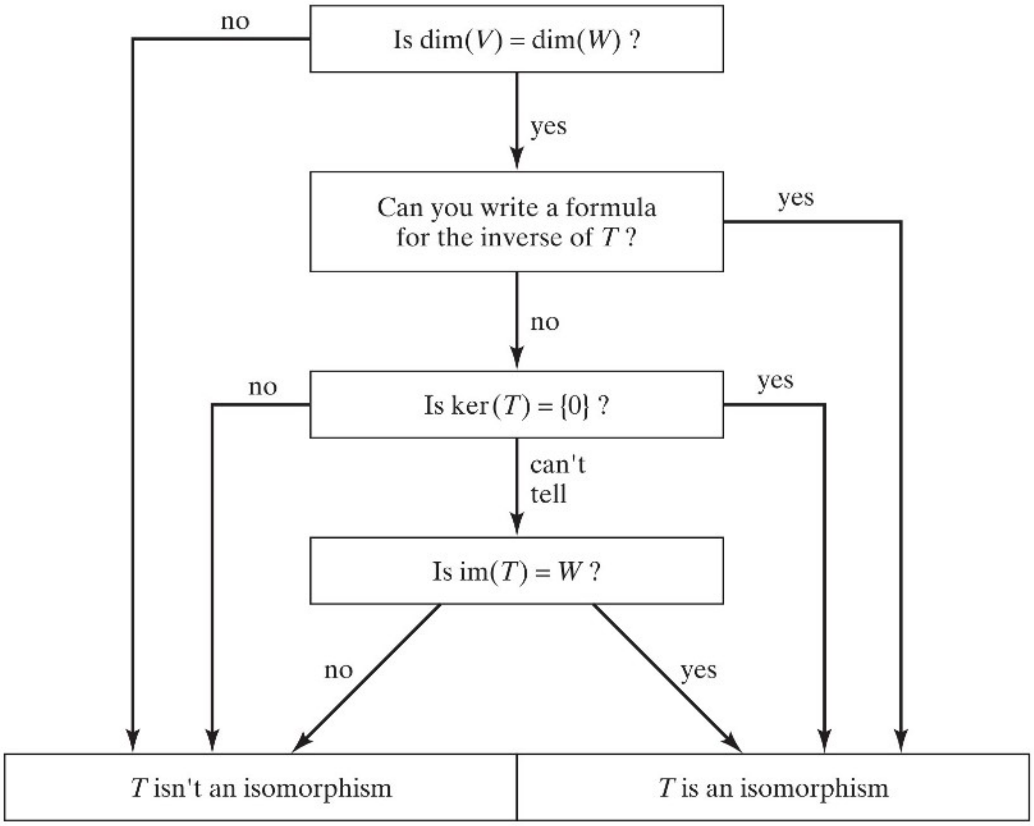

Figure 1 Is the linear transformation T from V to W an isomorphism? (V and W are finite dimensional linear spaces.)

Matrix of a linear transformation

Given two fin. dim. vector spaces V and W over $\mathbb{F}$ and a linear transformation $T: V \to W$, we can consider

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "L_2", "\cong"'] \\ \mathbb{F}^n \arrow[u, "L_1^{-1}"', "\cong"] \arrow[r, "A"] & \mathbb{F}^m \end{tikzcd}$$

Where $A(\vec{x}) := L_2(T(L_1^{-1}(\vec{x})))$, L_1 and L_2 are coordinate transformations. Since A is a composition of linear maps, it is itself a linear composition of linear maps, it is itself a linear map and thus is an $m \times n$ matrix. To find this matrix A, we have to compute what it does on vectors $\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}$ and collect these in columns. But

$$A\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = L_2 T L_1^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = L_2 T(v_1) = [T(v_1)]_{\mathcal{B}_W}$$

$$A\begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix} = [T(v_2)]_{\mathcal{B}_W}, \dots, A\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix} = [T(v_n)]_{\mathcal{B}_W}$$

where $\mathcal{B}_V = (v_1, v_2, \dots, v_n)$ is a V-basis corresponding to L_1, and $\mathcal{B}_W$ is a W-basis corresponding to L_2. In other words

$$A = \begin{pmatrix} | & & | \\ [T(v_1)]_{\mathcal{B}_W} & \cdots & [T(v_n)]_{\mathcal{B}_W} \\ | & & | \end{pmatrix}$$

A is called a $\mathcal{B}_V - \mathcal{B}_W$ matrix of linear map T. and

$$T(v) = L_2^{-1}\left(A \cdot \left(L_1(v)\right)\right)$$

for all $v \in V$.

Example: Let $V = \operatorname{span}(\cos(x), \sin(x)) \subseteq C^\infty$, Example: Let $V = \text{span}(\cos(x), \sin(x)) \subseteq C^\infty$, where C^∞ denotes all functions $\mathbb{R} \to \mathbb{R}$ that are infinitely differentiable. So, $V = \{a\cos x + b\sin x \mid a, b \in \mathbb{R}\}$. Consider the transformation $T: V \to V$ given by $T(f) = 3f + 2f' - f''$. We know that T is linear. We would like to find matrix A of T with respect to basis $(\cos x, \sin x)$. Now, on basis elements

$$
\begin{aligned}
T(\cos x) &= 3\cos x - 2\sin x + \cos x \\
&= 4\cos x - 2\sin x \\
T(\sin x) &= 3\sin x + 2\cos x + \sin x \\
&= 2\cos x + 4\sin x
\end{aligned}
$$

So, we have that

$$
A = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cc}
T(\cos x) & T(\sin x) \\
\end{array}
$$

$$
A = \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} \begin{array}{l} \cos x \\ \sin x \end{array}
$$

Q: Is T an isomorphism? Since $T = L_1 A \cdot L_1^{-1}$ and L_1 is an isomorphism, then T is an isomorphism if and only if A is, which happens if and only if matrix A is invertible, which happens if and only if $\det A \neq 0$.

So, $\det A = 4 \cdot 4 + 2 \cdot 2 = 20 \neq 0$. Therfore

So, $\det A = 4 \cdot 4 + 2 \cdot 2 = 20 \neq 0$. Therfore T is an isomorphism.