Nine Chapters on the Mathematical Art by Jiuzhang Suanshu, more than 2000 years ago.

The yield of one bundle of inferior rice, two bundles of medium grade rice, and three bundles of superior rice is 39 *dou* of grain. The yield of one bundle of inferior rice, three bundles of medium grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium grain rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

x – One bundle of inferior rice

y – One bundle of medium grade rice

z – One bundle of superior grade rice

Following text, we get a system of linear equations:

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases} \xrightarrow[\text{need to transform}]{\text{to solve this, we}} \begin{cases} x = \cdots \\ y = \cdots \\ z = \cdots \end{cases}$$

$$\left\{ 3x + 2y + z = 26 \right. \qquad\qquad \left\{ z = \dots \right.$$

So, we eliminate terms that are off the diagonal and make the coefficients of the variables along the diagonal equal to 1

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases}$$

- eliminate these

- put '1' here.

We accomplish this step-by-step

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases} \xrightarrow{-1^{st} \text{ equation}} \begin{cases} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{cases}$$

To eliminate the variable x from the third equation, we subtract the first equation from the third equation three times

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$$3x + 6y + 9z = 117 \quad (3 \times 1^{st} \text{ equation})$$

$$\begin{cases} x + 2y + 3z = 39 \\ \quad\;\; y - z = -5 \\ 3x + 2y + z = 26 \quad -3 \times 1^{st} \text{ equation} \end{cases} \longrightarrow \begin{cases} x + 2y + 3z = 39 \\ \quad\;\; y - z = -5 \\ \;\; -4y - 8z = -91 \end{cases}$$

Similarly, to eliminate y above and below the diagonal:

$$\begin{cases} x + 2y + 3z = 39 \quad -2 \times 2^{nd} \text{ equation} \\ \quad\;\; y - z = -5 \\ \;\; -4y - 8z = -91 \quad +4 \times 2^{nd} \text{ equation} \end{cases} \longrightarrow \begin{cases} x \qquad + 5z = 49 \\ \quad y - z = -5 \\ \qquad -12z = -111 \end{cases}$$

Befor eliminating z above the diagonal, we make the coefficient of z on the diagonal equal to 1, be dividing the last equation by -12.

Before eliminating z above the diagonal, we make the coefficient of z on the diagonal equal to 1, be dividing the last equation by -12.

$$\begin{cases} x \qquad + 5z = 49 \\ \qquad y - z = -5 \\ \qquad -12z = -111 \end{cases} \xrightarrow[\div(-12)]{} \begin{cases} x \qquad + 5z = 49 \\ \qquad y - z = -5 \\ \qquad\quad z = 9.25 \end{cases}$$

Finally, we eliminate the variable z above the diagonal:

$$\begin{cases} x \qquad + 5z = 49 & -5\times \text{third equation} \\ \qquad y - z = -5 & + \text{third equation} \\ \qquad\quad z = 9.25 \end{cases} \longrightarrow \begin{cases} x = 2.75 \\ y = 4.25 \\ z = 9.25 \end{cases}$$

What does this mean? Substituting:

$$2.75 + 2 \times 4.25 + 3 \times 9.25 = 39$$
$$2.75 + 3 \times 4.25 + 2 \times 9.25 = 34$$
$$3 \times 2.75 + 2 \times 4.25 + 9.25 = 26.$$

What we did here is basically Gaussian Elimination.

The Geometry of Linear Equations

Any list of numbers $(s_1, s_2, \cdots, s_n)$ can be thought of as a point in n-dimensional space, called a real vector space. We denote that vector space by R^n

So if we are considering linear equations with n unknowns, the solutions are points in R^n

Now, any linear equation defines a point set with dimension one less than the space. For example:

- if we are in 2-space (2 unknowns), a linear equation defines a line.
- if we are in 3-space (3 unknowns), a linear equation defines a plane.
- in higher dimensions, we refer to all such sets as *hyperplanes.*

Question: why does a linear equation define a point-set of dimension one less than the space?

Examples in $\mathbb{R}^2$

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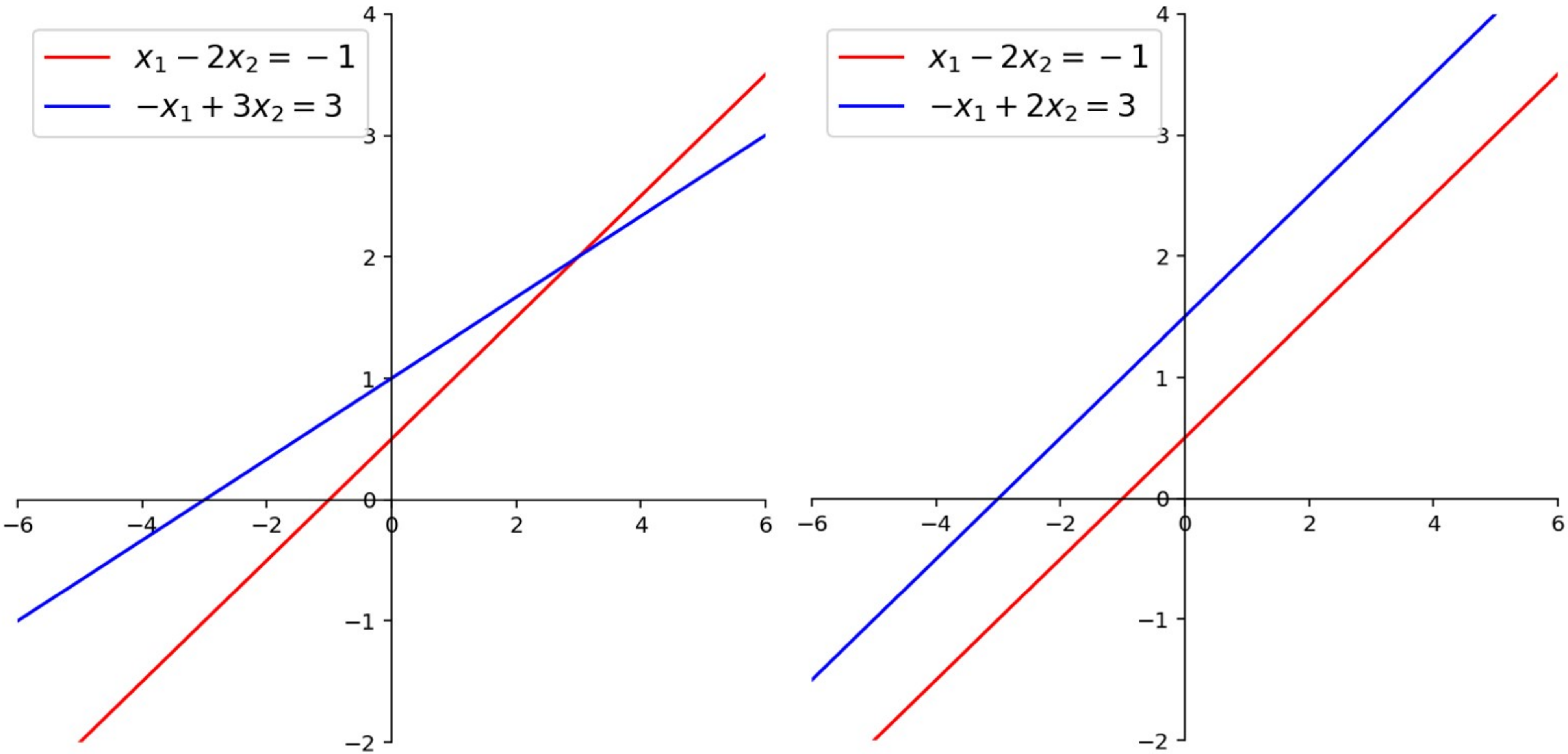
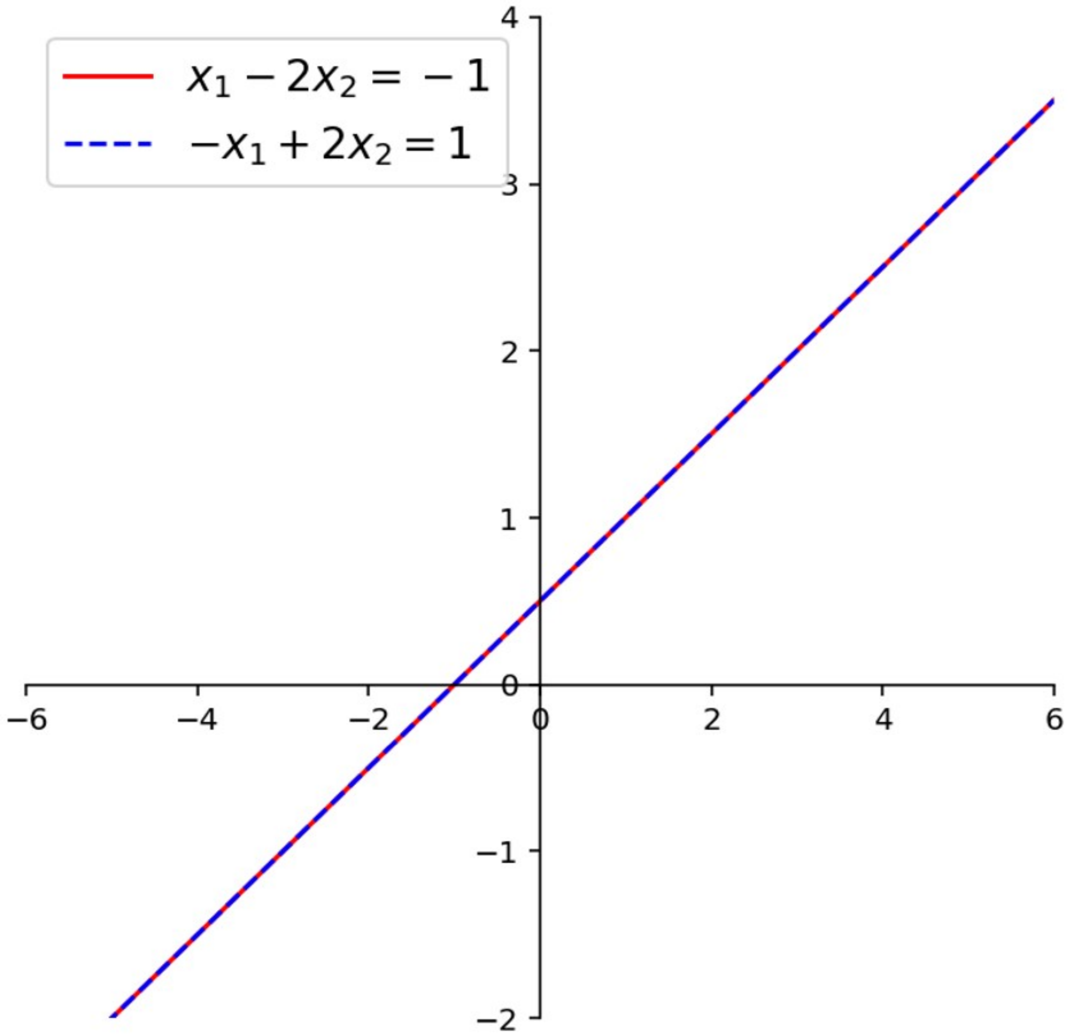




$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$ One solution

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_3 = 3 \end{cases}$$ No solution

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$ Infinitely many solutions

We clearly see that these are all the possibilities in $\mathbb{R}^2$. But what can happen in $\mathbb{R}^3$?

We had a single solution in our Chinese example.

Some possibilities in $\mathbb{R}^3$:

Figure 1.1

Example 1

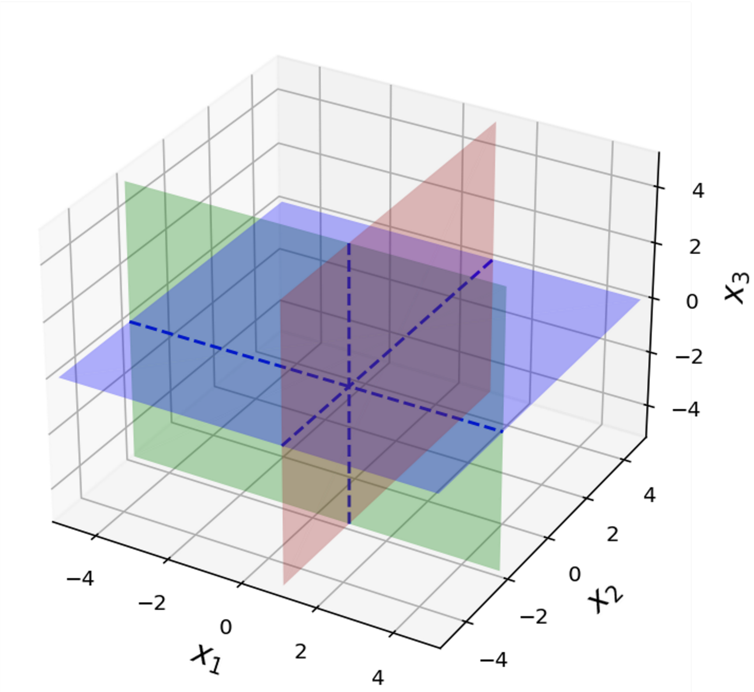


Figure 1.2

Example 2

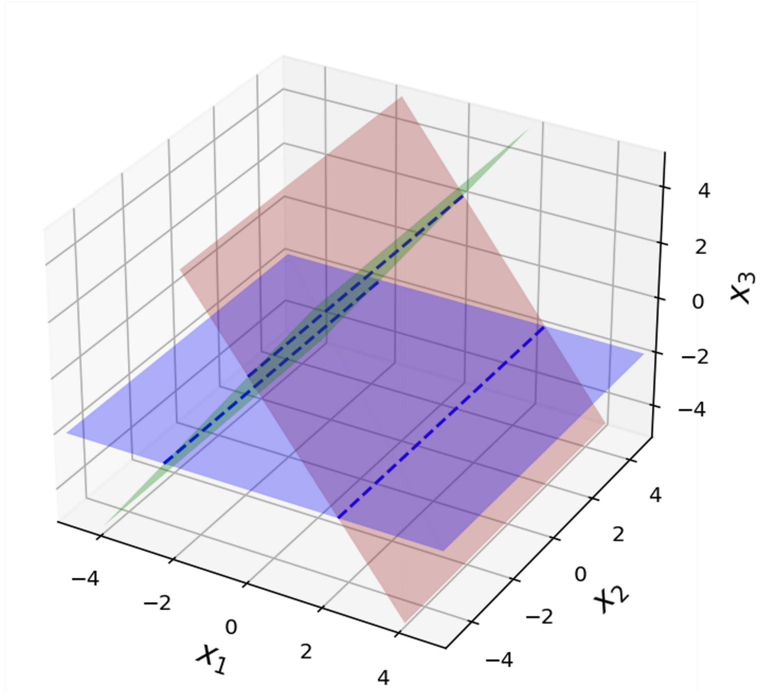


Figure 1.3

Example 3

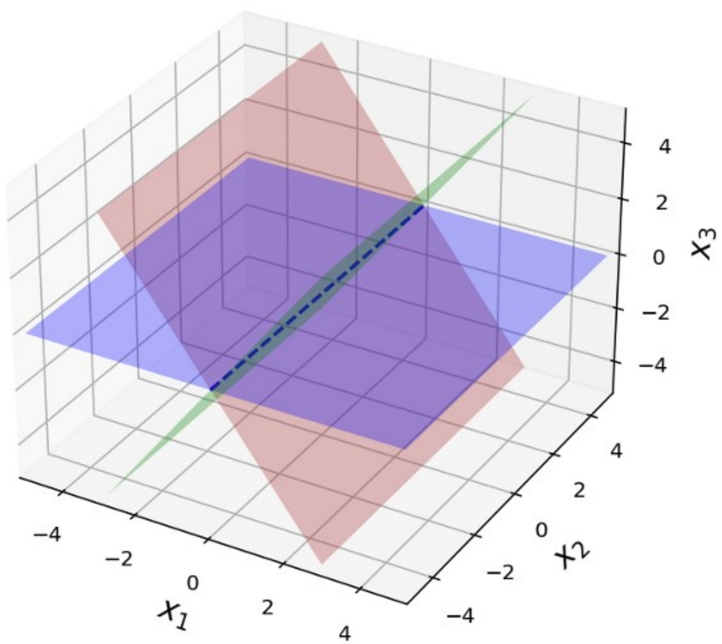
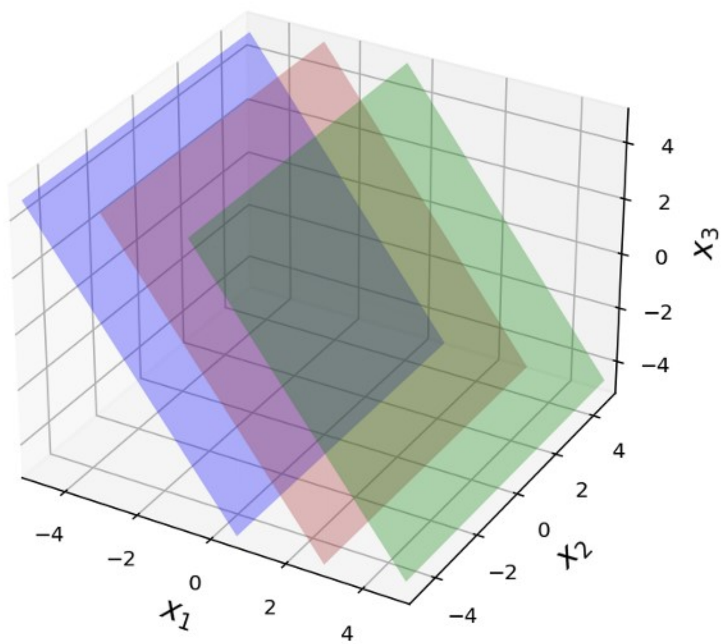


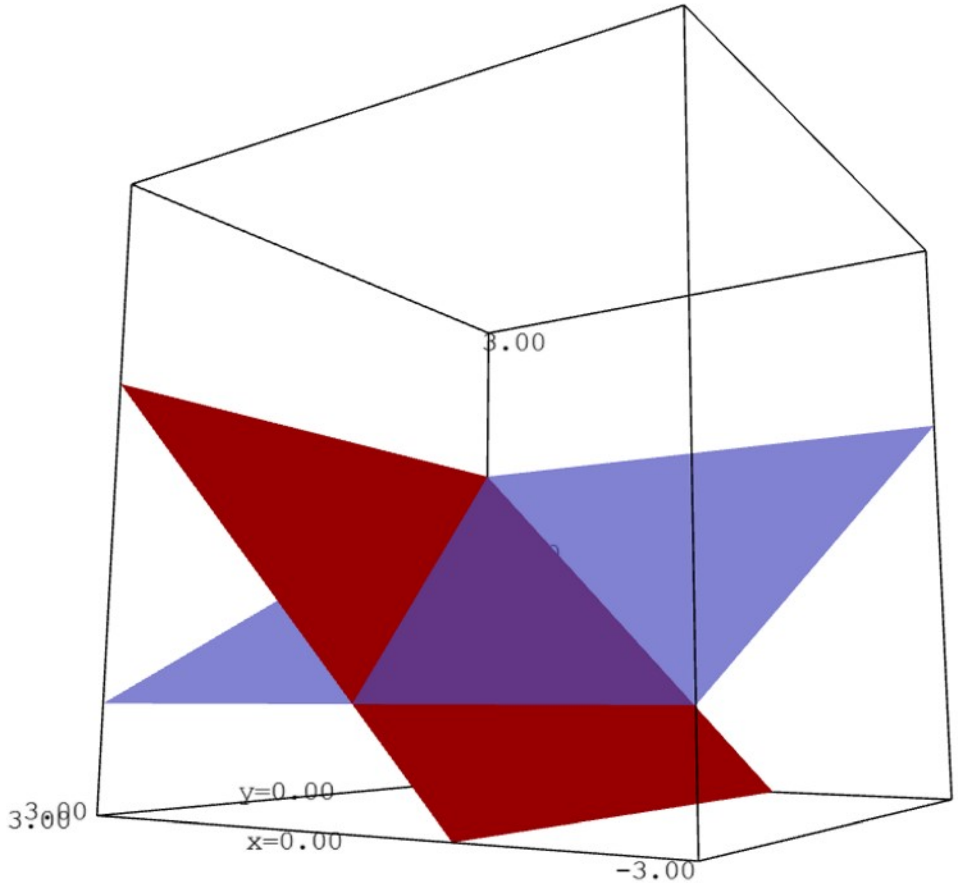
Figure 1.4

Example 4



A System with Infinitely many Solutions

$$\begin{cases} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 9 \end{cases} \xrightarrow{\text{after elimination}} \begin{cases} x - \phantom{y +{}} z = 2 \\ \phantom{x -{}} y + 2z = -1 \\ 0 = 0 \end{cases}$$



After neglecting $0=0$, the two planes intersect on a line. This system has infinitely many solutions.

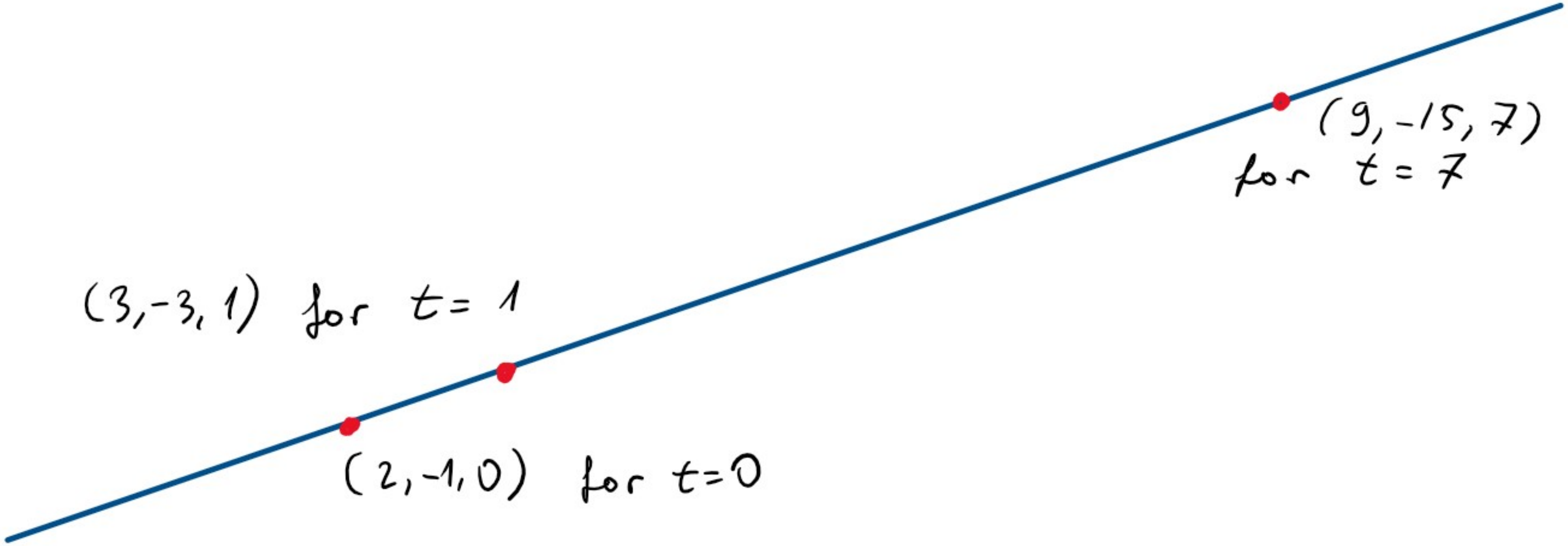
We can rewrite this two equations as

$$\begin{cases} x = 2 + z \\ y = -1 - 2z \end{cases}$$

By choosing z, we determine x and y and get a solution. For example:

- Choose $z=1$: Then $x = 2+z = 2+1 = 3$ and $y = -1-2z = -1-2 = -3$. The solution is $(x,y,z) = (3,-3,1)$
- Choose $z=7$: Then $x=9$ and $y=-15$. $(x,y,z)=(9,-15,7)$

- Choose $z = 7$: Then $x = 9$ and $y = -15$. $(x, y, z) = (9, -15, 7)$
- In general $z = t$: Then $x = 2 + t$; $y = -1 - 2t$ and
  $(x, y, z) = (2+t, -1-2t, t)$ $\rightsquigarrow$ Equation of a line



A System without Solutions

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{cases} \xrightarrow{\text{after elimination}} \begin{cases} x \quad\quad - z = 2 \\ \quad y + 2z = -1 \\ \quad\quad\quad 0 = -6 \end{cases}$$

Whatever values we choose for (x, y, z) the equation $0 = -6$

Whatever values we choose for (x, y, z) the equation $0 = -6$ will never be satisfied. The system is inconsistent, that is, it has no solutions.

(2)

We solved the previous system by only working with highlited numbers.

$$\begin{cases} 1x + 2y + 3z = 39 \\ 1x + 3y + 2z = 34 \\ 3x + 2y + 1z = 26 \end{cases},$$

So, we only need to look at a rectengular arrey of numbers

3×4 Matrix

$$\begin{array}{c} 4\ \cancel{\text{rows}}\ \text{columns} \\ \begin{pmatrix} 1 & 2 & 3 & 39 \\ 1 & 3 & 2 & 34 \\ 3 & 2 & 1 & 26 \end{pmatrix} \end{array} \quad 3\ \cancel{\text{columns}}\ \text{rows}$$

This is called a **matrix**. This matrix has 3 rows and 4 columns, so it is **3×4** matrix. Usually matrix entries

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

are labelled by double subscripts The **first** refers to the **row** and **second** to the **column**.

③ entry a_{ij} is located on i^{th} row and j^{th} column.

Equality of matrices: $A = B \iff a_{ij} = b_{ij}$ and A, B same size

A is $n \times n \iff A$ is a square matrix.

$a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ form a main diagonal

A is diagonal $\iff a_{ij} = 0$ if $i \neq j$

A is upper triangular $\iff a_{ij} = 0$ when $i > j$

$A = 0$ is a zero matrix $\iff a_{ij} = 0$ for all i, j.

Note different notation for parenthesis

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

B, C, D, E – square

C – diagonal

C, D – upper triang.

C, E – lower triang.

(4) Matrices with only one row or column are of particular interest.

Vectors and vector spaces

A matrix with only one column is called a column vector, or simply a vector. The entries of a vector are called its components. The set of all column vectors with n components is denoted by $\mathbb{R}^n$; we will refer to $\mathbb{R}^n$ as a *vector space*.

A matrix with only one row is called a row vector.

Examples: $\begin{pmatrix} 1 \\ 2 \\ 9 \\ 1 \end{pmatrix} \in \mathbb{R}^4$ is a (column) vector; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$;

$(1\ 5\ 5\ 3\ 7)$ row vector with five components;

m columns of $n \times m$ matrix are vectors in $\mathbb{R}^n$.

Geomertic point of view: represent a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ in a Cartesian coordinate plane as arrow from the origin to the point (x, y). Similarly in $\mathbb{R}^3$.

Note the arrow denoting a vector

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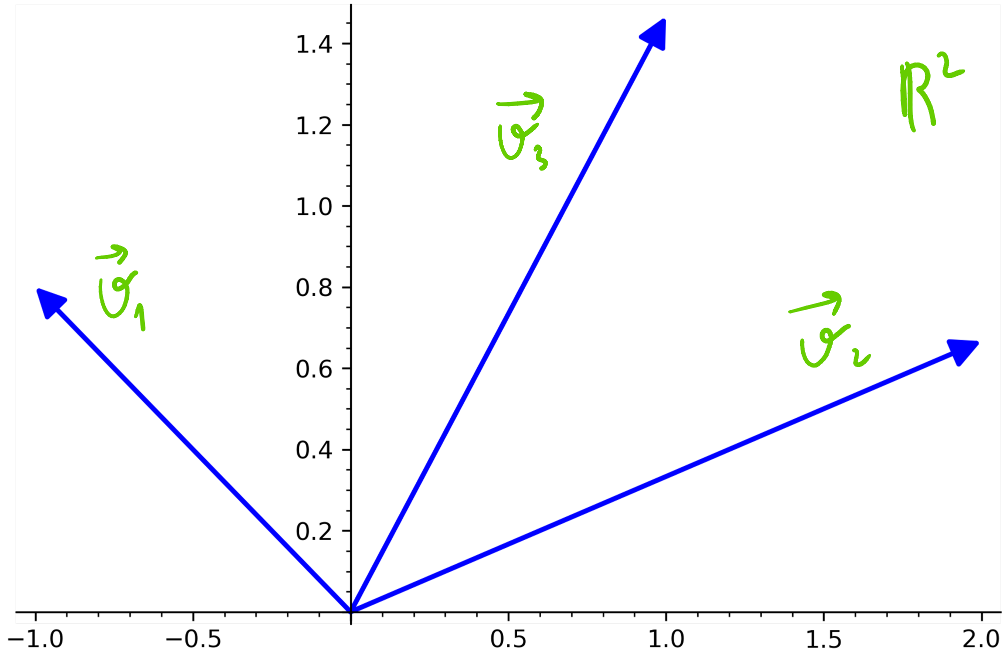
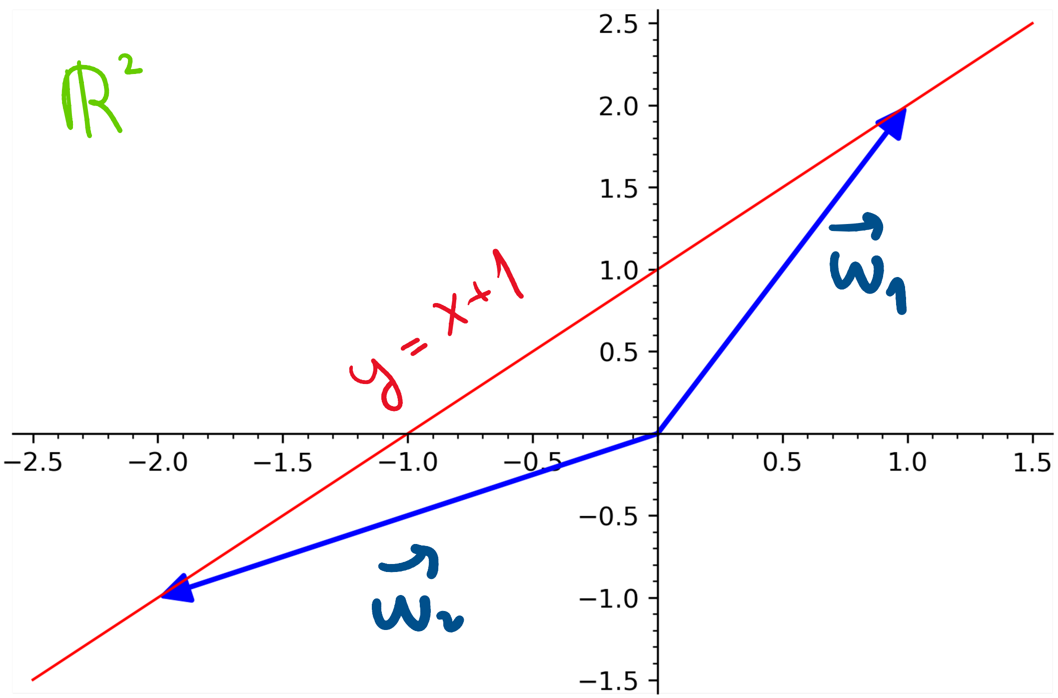


$\vec{v}_1 = \begin{pmatrix} -1 \\ 0.8 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 0.6 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 1 \\ 1.4 \end{pmatrix}$



$\vec{w}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} x \\ x+1 \end{pmatrix}$ where x is arbitrary real number.

(6)

$$\begin{cases} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{cases} \xrightarrow[\text{matrix}]{\text{Augmented}} \left(\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right)$$

$$\begin{pmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{pmatrix}$$ - coefficient matrix

$$\boxed{\text{To solve the system, we perform all the operations for elimination on an augmented matrix}}$$

Two approaches are equivalent, for example, dividing the first equation by 2 is the same as dividing the first row of the augmented matrix by 2.

$$2x + 8y + 4z = 2 \xrightarrow{\div 2} x + 4y + 2z = 1 \Longleftrightarrow \left(\begin{array}{ccc|c} 2 & 8 & 4 & 2 \end{array}\right) \xrightarrow{\div 2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \end{array}\right)$$

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$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -2\,(\mathrm{I}) \\ -4\,(\mathrm{I}) \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} \\ \div(-3) \\ \\ \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} -4\,(\mathrm{II}) \\ \\ +6\,(\mathrm{II}) \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right] \begin{array}{l} \\ \\ \div(-3) \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} +2\,(\mathrm{III}) \\ -\,(\mathrm{III}) \\ \\ \end{array}$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left|\begin{array}{rrrr} 2x + & 8y + & 4z = & 2 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \\ -2\,(\mathrm{I}) \\ -4\,(\mathrm{I}) \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & -3y - & 3z = & 3 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} \\ \div(-3) \\ \\ \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & y + & z = & -1 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} -4\,(\mathrm{II}) \\ \\ +6\,(\mathrm{II}) \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & -3z = & -9 \end{array}\right| \begin{array}{l} \\ \\ \div(-3) \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & z = & 3 \end{array}\right| \begin{array}{l} +2\,(\mathrm{III}) \\ -\,(\mathrm{III}) \\ \\ \end{array}$$
$$\downarrow$$
$$\left|\begin{array}{rrrr} x & & = & 11 \\ & y & = & -4 \\ & & z = & 3 \end{array}\right|.$$

The solution is often represented by a vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}$$

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$$\begin{cases} x_1 - x_2 + 4x_5 = 2 \\ x_3 - x_5 = 2 \\ x_4 - x_5 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 2 + x_2 - 4x_5 \\ x_3 = 2 + x_5 \\ x_4 = 3 + x_5 \end{cases}$$

Choose $x_2 = t$, $x_5 = r$, then $x_1 = 2 + t - 4r$; $x_3 = 2 + r$; $x_4 = 3 + r$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 + t - 4r \\ t \\ 2 + r \\ 3 + r \\ r \end{pmatrix}$$

Why was this so easy to solve?

- P1: The leading coefficient in each equation is 1. (The leading coefficient is the coefficient of the leading variable.)
- P2: The leading variable in each equation does not appear in any of the other equations. (For example, the leading variable x_3 of the second equation appears neither in the first nor in the third equation.)
- P3: The leading variables appear in the "natural order," with increasing indices as we go down the system (x_1, x_3, x_4 as opposed to x_3, x_1, x_4, for example).

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$$\begin{cases} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{cases}$$

We wish to reduce this system to the system with properties P1, P2 and P3. The reduced system will then be easy to solve. We proceed equation by equation from top to bottom.

$$\left|\begin{array}{l} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array}\right| \div 2 \qquad \left[\begin{array}{ccccc:c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \div 2$$

$$\downarrow$$

$$\left|\begin{array}{l} x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 1 \\ x_1 + 2x_2 - x_3 + 2x_4 = 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9 \end{array}\right| \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array} \qquad \left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array}$$

$$\downarrow$$

$$\left|\begin{array}{l} x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 1 \\ x_4 - 2x_5 = 3 \\ x_3 - 2x_4 + 3x_5 = -2 \\ x_3 - x_5 = 4 \end{array}\right| \qquad \left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]$$

Now we have properties P1 and P2.

Not P3.

[redacted]. To satisfy this requirement, we will swap the second equation with the third equation. (In the following summary, we will specify when such a swap is indicated and how it is to be performed.)

Then we can eliminate x_3 from the first and fourth equations.

$$\left| \begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ x_3 \qquad - x_5 &=& 4 \end{array} \right| \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array} \quad \left[\begin{array}{rrrrr:r} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array}$$

$$\downarrow$$

$$\left| \begin{array}{rcl} x_1 + 2x_2 \qquad - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array} \right| \quad \left[\begin{array}{rrrrr:r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array} \right]$$

Now we turn our attention to the third equation, with leading variable x_4. We need to eliminate x_4 from the other three equations.

$$\left| \begin{array}{rcl} x_1 + 2x_2 \qquad - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 \quad 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array} \right| \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array} \quad \left[\begin{array}{rrrrr:r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array} \right] \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array}$$

$$\downarrow$$

$$\left| \begin{array}{rcl} x_1 + 2x_2 \qquad + 3x_5 &=& 2 \\ x_3 \qquad - x_5 &=& 4 \\ x_4 - 2x_5 &=& 3 \\ 0 &=& 0 \end{array} \right| \quad \left[\begin{array}{rrrrr:r} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$x_1 = 2 - 2x_2 - 3x_5$

$x_3 = 4 + x_5$

$x_4 = 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2-2t-3r \\ t \\ 4+r \\ 3+2r \\ r \end{pmatrix}$$

as a vector

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Solving a system of linear equations

We proceed from equation to equation, from top to bottom.

Suppose we get to the ith equation. Let x_j be the leading variable of the system consisting of the ith and all the subsequent equations. (If no variables are left in this system, then the process comes to an end.)

- If x_j does not appear in the ith equation, swap the ith equation with the first equation below that does contain x_j.
- Suppose the coefficient of x_j in the ith equation is c; thus this equation is of the form $cx_j + \cdots = \cdots$. Divide the ith equation by c.
- Eliminate x_j from all the other equations, above and below the ith, by subtracting suitable multiples of the ith equation from the others.

Now proceed to the next equation.

If an equation *zero* = *nonzero* emerges in this process, then the system fails to have solutions; the system is *inconsistent*.

When you are through without encountering an inconsistency, solve each equation for its leading variable. You may choose the nonleading variables freely; the leading variables are then determined by these choices.

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Reduced row-echelon form

A matrix is in reduced row-echelon form if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is a 1, called the *leading* 1 (or *pivot*) in this row.

b. If a column contains a leading 1, then all the other entries in that column are 0.

c. If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Condition c implies that rows of 0's, if any, appear at the bottom of the matrix.

Conditions a, b, and c defining the reduced row-echelon form correspond to the conditions P1, P2, and P3 that we imposed on the system.

Note that the leading 1's in the matrix

$$E = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

correspond to the leading variables in the reduced system,

$$\left| \begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & + & 3x_5 & = & 2 \\ & & & x_3 & & - & x_5 & = & 4 \\ & & & & x_4 & - & 2x_5 & = & 3 \end{array} \right| .$$

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The operations we perform when bringing a matrix into reduced row-echelon form are referred to as elementary row operations. Let's review the three types of such operations.

Types of elementary row operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.